Introductory Mathematics
Illustrative Problems and Solutions
(Part 1 of 3)

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1. Integers and rational numbers

Computation with rational numbers – Questions 1/2

$$\frac{4}{10} \times \frac{4}{5} - \frac{3}{5} \div \frac{5}{2} =$$

$$\frac{3}{2} \times \frac{1}{4} - \frac{2}{6} \div \frac{8}{3} =$$

$$\frac{5}{4} \times \frac{4}{8} - \frac{3}{7} \div \frac{48}{14} =$$

$$\left(\frac{7}{5} - \frac{2}{3}\right) \div \frac{2}{5} =$$

$$\left(\frac{2}{3} - \frac{1}{7}\right) \div \frac{33}{18} =$$

$$\left(\frac{5}{8} - \frac{2}{5}\right) \div \frac{11}{4} =$$

Computation with rational numbers – Questions 2/2

$$\frac{2}{5} \div \frac{4}{3} + \frac{5}{2} \times \frac{1}{3} =$$

$$\frac{1}{4} \div \frac{2}{3} + \frac{1}{2} \times \frac{1}{3} =$$

$$\frac{5}{2} \div \frac{1}{3} - \frac{7}{2} \times \frac{1}{3} =$$

$$\frac{5}{2} \div \frac{4}{3} - \frac{3}{2} \div \frac{4}{3} =$$

$$\left(\frac{3}{2} - \frac{1}{3}\right) \div \left(\frac{3}{4} + \frac{4}{3}\right) =$$

$$\left(\frac{3}{2} - \frac{1}{3}\right) \div \left(\frac{3}{2} + \frac{2}{3}\right) =$$

$$\left(\frac{2}{3} - \frac{1}{7}\right) \div \frac{11}{9} =$$

Computation with rational numbers – Answers 1/3

$$\frac{4}{10}\times\frac{4}{5}-\frac{3}{5}\div\frac{5}{2} = \frac{16}{50}-\frac{3}{5}\times\frac{2}{5}=\frac{16}{50}-\frac{6}{25}=\frac{16-12}{50}=\frac{4}{50}=\frac{2}{25}$$

$$\frac{3}{2}\times\frac{1}{4}-\frac{2}{6}\div\frac{8}{3} = \frac{3}{8}-\frac{2}{6}\times\frac{3}{8}=\frac{3}{8}-\frac{1}{8}=\frac{1}{4}$$

$$\frac{5}{4}\times\frac{4}{8}-\frac{3}{7}\div\frac{48}{14} = \frac{5}{8}-\frac{3}{7}\times\frac{14}{48}=\frac{5}{8}-\frac{1}{8}=\frac{1}{2}$$

$$\left(\frac{7}{5}-\frac{2}{3}\right)\div\frac{2}{5} = \frac{21-10}{15}\times\frac{5}{2}=\frac{11}{15}\times\frac{5}{2}=\frac{11}{6}$$

$$\left(\frac{2}{3}-\frac{1}{7}\right)\div\frac{33}{18} = \frac{14-3}{21}\times\frac{18}{33}=\frac{11}{21}\times\frac{18}{33}=\frac{2}{7}$$

Computation with rational numbers – Answers 2/3

$$\left(\frac{5}{8}-\frac{2}{5}\right)\div\frac{11}{4} = \frac{25-16}{40}\times\frac{4}{11}=\frac{9}{40}\times\frac{4}{11}=\frac{9}{110}$$

$$\frac{2}{5}\div\frac{4}{3}+\frac{5}{2}\times\frac{1}{3} = \frac{2}{5}\times\frac{3}{4}+\frac{5}{6}=\frac{3}{10}+\frac{5}{6}=\frac{9}{30}+\frac{25}{30}=\frac{34}{30}=\frac{17}{15}$$

$$\frac{1}{4}\div\frac{2}{3}+\frac{1}{2}\times\frac{1}{3} = \frac{1}{4}\times\frac{3}{2}+\frac{1}{6}=\frac{3}{8}+\frac{1}{6}=\frac{9}{24}+\frac{4}{24}=\frac{13}{24}$$

$$\frac{5}{2}\div\frac{1}{3}-\frac{7}{2}\times\frac{1}{3} = \frac{5}{2}\times\frac{3}{1}-\frac{7}{6}=\frac{15}{2}-\frac{7}{6}=\frac{45}{6}-\frac{7}{6}=\frac{38}{6}=\frac{19}{3}$$

$$\frac{5}{2}\div\frac{4}{3}-\frac{3}{2}\div\frac{4}{3} = \frac{5}{2}\times\frac{3}{4}-\frac{3}{2}\times\frac{3}{4}=1\times\frac{3}{4}=\frac{3}{4}$$

Computation with rational numbers – Answers 3/3

$$\left(\frac{3}{2}-\frac{1}{3}\right)\div\left(\frac{3}{4}+\frac{4}{3}\right) = \left(\frac{9}{6}-\frac{2}{6}\right)\div\left(\frac{9}{12}+\frac{16}{12}\right)=\frac{7}{6}\div\frac{25}{12}=\frac{7}{6}\times\frac{12}{25}=\frac{14}{25}$$

$$\left(\frac{3}{2}-\frac{1}{3}\right)\div\left(\frac{3}{2}+\frac{2}{3}\right) = \left(\frac{9}{6}-\frac{2}{6}\right)\div\left(\frac{9}{6}+\frac{4}{6}\right)=\frac{7}{6}\div\frac{13}{6}=\frac{7}{6}\times\frac{6}{13}=\frac{7}{13}$$

$$\left(\frac{2}{3}-\frac{1}{7}\right)\div\frac{11}{9} = \left(\frac{14}{21}-\frac{3}{21}\right)\div\frac{11}{9}=\frac{11}{21}\div\frac{11}{9}=\frac{11}{21}\times\frac{9}{11}=\frac{3}{7}$$

GCD computations – example 1

Question

Find the greatest common divisor of the integers

$$440, \quad 352, \quad 231.$$

First, compute $\gcd(440, 352)$ using the Euclidean algorithm:

$$440 = 1 \times 352 + 88, \quad 352 = 4 \times 88 + 0 \Longrightarrow \gcd(440, 352) = 88$$

Next, compute $\gcd(88, 231)$ using the Euclidean algorithm:

$$\begin{aligned} 231 &= 2 \times 88 + 55 \\ 88 &= 1 \times 55 + 33 \\ 55 &= 1 \times 33 + 22 \\ 33 &= 1 \times 22 + 11 \\ 22 &= 2 \times 11 + 0 \\ &\Longrightarrow \gcd(88, 231) = 11 \end{aligned}$$

GCD computations – example 1

Question

Find the greatest common divisor of the integers

$$440, \quad 352, \quad 231.$$

Alternatively, we compute the prime factorization of each number:

$$440 = 2^3 \times 5 \times 11$$
$$352 = 2^5 \times 11$$
$$231 = 3 \times 7 \times 11$$

The common prime factor among all three numbers is 11. Therefore,

$$\gcd(440, 352, 231) = 11$$

GCD computations – example 2

Question

Find the greatest common divisor of the integers

$$312, \quad 455, \quad 429.$$

First, use the Euclidean algorithm to find $\gcd(312, 455)$:

$$\begin{aligned} 455 &= 1 \times 312 + 143 \\ 312 &= 2 \times 143 + 26 \\ 143 &= 5 \times 26 + 13 \\ 26 &= 2 \times 13 + 0 \end{aligned}$$

So, $\gcd(312, 455) = 13$.
Now find $\gcd(13, 429)$:

$$429 = 33 \times 13 + 0$$

Thus, $\gcd(312, 455, 429) = 13$.

GCD computations – example 2

Question

Find the greatest common divisor of the integers

$$312, \quad 455, \quad 429.$$

Alternatively, we compute the prime factorizations:

$$312 = 2^3 \times 3 \times 13$$
$$455 = 5 \times 7 \times 13$$
$$429 = 3 \times 11 \times 13$$

The only common prime factor is 13, so:

$$\gcd(312, 455, 429) = 13$$

GCD computations – example 3

Question

Find the greatest common divisor of the integers

$$374, \quad 340, \quad 357.$$

First, use the Euclidean algorithm to find $\gcd(374, 340)$:

$$374 = 1 \times 340 + 34$$
$$340 = 10 \times 34 + 0$$

So, $\gcd(374, 340) = 34$.
Now find $\gcd(34, 357)$:

$$357 = 10 \times 34 + 17$$
$$34 = 2 \times 17 + 0$$

Thus, $\gcd(374, 340, 357) = 17$.

GCD computations – example 3

Question

Find the greatest common divisor of the integers

$$374, \quad 340, \quad 357.$$

We compute the prime factorizations:

$$\begin{aligned} 374 &= 2 \times 11 \times 17 \\ 340 &= 2^2 \times 5 \times 17 \\ 357 &= 3 \times 7 \times 17 \end{aligned}$$

The only common prime factor is 17, so:

$$\gcd(374, 340, 357) = 17$$

GCD computations – example 4

Question

Find the greatest common divisor of the integers

$$156, \quad 182, \quad 429.$$

First, use the Euclidean algorithm to find $\gcd(156, 182)$:

$$182 = 1 \times 156 + 26$$
$$156 = 6 \times 26 + 0$$

So, $\gcd(156, 182) = 26$.
Now find $\gcd(26, 429)$:

$$429 = 16 \times 26 + 13$$
$$26 = 2 \times 13 + 0$$

Thus, $\gcd(156, 182, 429) = 13$.

GCD computations – example 4

Question

Find the greatest common divisor of the integers

$$156, \quad 182, \quad 429.$$

We compute the prime factorizations:

$$\begin{aligned} 156 &= 2^2 \times 3 \times 13 \\ 182 &= 2 \times 7 \times 13 \\ 429 &= 3 \times 11 \times 13 \end{aligned}$$

The only common prime factor is 13, so:

$$\gcd(156, 182, 429) = 13$$

Smallest factor and GCD – example 1

Question

Find the sum of the greatest common divisor (GCD) and the smallest common prime factor of the numbers

$$170, \quad 595, \quad 255.$$

We begin with the prime factorizations:

$$170 = 2 \times 5 \times 17$$
$$595 = 5 \times 7 \times 17$$
$$255 = 3 \times 5 \times 17$$

The common prime factors are $5,\ 17$. So, the greatest common divisor is:

$$\gcd(170, 595, 255) = 5 \times 17 = 85$$

The smallest positive common prime factor is 5. Thus, the sum is:

$$85 + 5 = 90$$

Smallest factor and GCD – example 2

Question

Find the sum of the greatest common divisor (GCD) and the smallest common prime factor of the numbers

$$170, \quad 595, \quad 340.$$

We begin with the prime factorizations:

$$170 = 2 \times 5 \times 17$$
$$595 = 5 \times 7 \times 17$$
$$340 = 2^2 \times 5 \times 17$$

The common prime factors are $5,\ 17.$ So, the greatest common divisor is:

$$\gcd(170, 595, 340) = 5 \times 17 = 85$$

The smallest common prime factor is $5.$ Thus, the sum is:

$$85 + 5 = 90$$

2. Expression simplification

Expression simplification – example 1

Question

Simplify

$$\left(\frac{\frac{a^2}{b^2}-\frac{a}{b}}{\frac{a^2+b^2}{ab}-2}\right)\div\frac{a^2}{b^2}, \quad \text{where } a\neq 0, b\neq 0, a\neq b$$

$$\begin{aligned}\left(\frac{\frac{a^2}{b^2}-\frac{a}{b}}{\frac{a^2+b^2}{ab}-2}\right)\div\frac{a^2}{b^2} &= \left(\frac{\frac{a^2-ab}{b^2}}{\frac{a^2+b^2-2ab}{ab}}\right)\times\frac{b^2}{a^2}=\left(\frac{\frac{a(a-b)}{b}}{\frac{(a-b)^2}{a}}\right)\times\frac{b^2}{a^2}\\ &= \frac{a^2}{b(a-b)}\times\frac{b^2}{a^2}=\frac{b}{a-b}\end{aligned}$$

Expression simplification – example 2

Question

Simplify

$$\left(b+\frac{a-b}{1+ab}\right)\div\left(1-\frac{b(a-b)}{1+ab}\right), \quad \text{where } 1+ab\neq 0$$

$$\begin{aligned}\left(b+\frac{a-b}{1+ab}\right)\div\left(1-\frac{b(a-b)}{1+ab}\right) &= \frac{b+ab^2+a-b}{1+ab}\times\frac{1+ab}{1+ab-ab+b^2}\\ &= \frac{a(1+b^2)}{1+b^2}=a\end{aligned}$$

Expression simplification – example 3

Question

Evaluate

$$\frac{3}{1+\frac{1}{5+3\sqrt{3}}}-1$$

$$\frac{3}{1+\frac{1}{5+3\sqrt{3}}}-1 \quad = \quad \frac{3(5+3\sqrt{3})}{6+3\sqrt{3}}-1=\frac{15+9\sqrt{3}-6-3\sqrt{3}}{6+3\sqrt{3}}=\frac{9+6\sqrt{3}}{6+3\sqrt{3}}=\sqrt{3}$$

Expression simplification – example 4

Question

Simplify

$$\left(\frac{1}{1-x}-1\right)\div\left(x-\frac{1-2x^2}{1-x}+1\right), \quad \text{where } x\neq 1,\ x\neq 0$$

$$\begin{aligned}\left(\frac{1}{1-x}-1\right)\div\left(x-\frac{1-2x^2}{1-x}+1\right) &= \frac{1-1+x}{1-x}\div\frac{x-x^2-1+2x^2+1-x}{1-x}\\ &= \frac{x}{1-x}\times\frac{1-x}{x^2}=\frac{1}{x}\end{aligned}$$

Expression simplification – example 5

Question

Simplify

$$\frac{x^{-1}+y^{-1}}{x^{-1}-y^{-1}}-\frac{x^{-1}-y^{-1}}{x^{-1}+y^{-1}}, \quad \text{where } x \neq y,\ x \neq -y,\ x \neq 0,\ y \neq 0$$

$$\begin{aligned}\frac{x^{-1}+y^{-1}}{x^{-1}-y^{-1}}-\frac{x^{-1}-y^{-1}}{x^{-1}+y^{-1}} &= \frac{(x^{-2}+y^{-2}+2x^{-1}y^{-1})-(x^{-2}+y^{-2}-2x^{-1}y^{-1})}{x^{-2}-y^{-2}} \\ &= \frac{4x^{-1}y^{-1}}{x^{-2}-y^{-2}}=\frac{4xy}{y^2-x^2}\end{aligned}$$

Expression simplification – example 6

Question

Simplify

$$\frac{x^{\frac{1}{2}}+1}{x+x^{\frac{1}{2}}+1} \div \frac{1}{x^{\frac{3}{2}}-1}, \quad \text{where } x \geq 0,\ x \neq 1$$

$$\frac{x^{\frac{1}{2}}+1}{x+x^{\frac{1}{2}}+1} \div \frac{1}{x^{\frac{3}{2}}-1} \quad = \quad \frac{x^{\frac{1}{2}}+1}{x+x^{\frac{1}{2}}+1} \times (x^{\frac{1}{2}}-1)(x+x^{\frac{1}{2}}+1) = x-1$$

Expression simplification – example 7

Question

Evaluate

$$\frac{\sqrt{6}}{\sqrt{24}-\sqrt{6}-\sqrt{12}}$$

$$\begin{aligned}
\frac{\sqrt{6}}{\sqrt{24}-\sqrt{6}-\sqrt{12}} &= \frac{\sqrt{6}}{2\sqrt{6}-\sqrt{6}-2\sqrt{3}} = \frac{\sqrt{6}}{\sqrt{6}-2\sqrt{3}} = \frac{\sqrt{6}(\sqrt{6}+2\sqrt{3})}{(\sqrt{6}-2\sqrt{3})(\sqrt{6}+2\sqrt{3})} \\
&= \frac{6+6\sqrt{2}}{-6} = -1-\sqrt{2}
\end{aligned}$$

Expression simplification – example 8

Question

Evaluate

$$\left(\frac{1-\sqrt{2}}{\sqrt{2}}+\frac{\sqrt{2}}{1+\sqrt{2}}\right)^2$$

$$\begin{aligned}
&= \left(\frac{1}{\sqrt{2}}-1+\frac{\sqrt{2}(1-\sqrt{2})}{(1+\sqrt{2})(1-\sqrt{2})}\right)^2 = \left(\frac{1}{\sqrt{2}}-1+\frac{\sqrt{2}(1-\sqrt{2})}{1-2}\right)^2 \\
&= \left(\frac{1}{\sqrt{2}}-1-\sqrt{2}(1-\sqrt{2})\right)^2 = \left(\frac{1}{\sqrt{2}}-1-\sqrt{2}+2\right)^2 = \left(\frac{1}{\sqrt{2}}+1-\sqrt{2}\right)^2 \\
&= \left(\frac{1+\sqrt{2}-2}{\sqrt{2}}\right)^2 = \left(\frac{-1+\sqrt{2}}{\sqrt{2}}\right)^2 = \frac{3-2\sqrt{2}}{2}
\end{aligned}$$

Expression simplification – example 9

Question

Simplify

$$\left(\frac{x}{y}+\frac{y}{x}-2\right)^2 \div \frac{(x-y)^3}{(xy)^2} \quad \text{where } x, y \neq 0,\ x \neq y$$

$$\left(\frac{x}{y}+\frac{y}{x}-2\right)^2 \div \frac{(x-y)^3}{(xy)^2} = \left(\frac{(x-y)^2}{xy}\right)^2 \times \frac{(xy)^2}{(x-y)^3} = \frac{(x-y)^4}{(xy)^2} \times \frac{(xy)^2}{(x-y)^3} = x-y$$

Expression simplification – example 10

Question

Evaluate

$$\frac{1}{1+\frac{4}{4+4\sqrt{3}}}+1$$

$$\begin{aligned}\frac{1}{1+\frac{4}{4+4\sqrt{3}}}+1 &= \frac{1}{1+\frac{4}{4(1+\sqrt{3})}}+1=\frac{1}{1+\frac{1}{1+\sqrt{3}}}+1=\frac{1+\sqrt{3}}{1+\sqrt{3}+1}+1\\ &= \frac{(1+\sqrt{3})(2-\sqrt{3})}{(2+\sqrt{3})(2-\sqrt{3})}+1=\frac{-1+\sqrt{3}}{1}+1=\sqrt{3}\end{aligned}$$

Expression simplification – example 11

Question

Simplify

$$\frac{\frac{x}{2}-y}{\frac{x}{2}+y}-\frac{\frac{x}{2}+y}{\frac{x}{2}-y}, \quad \text{where } x \neq 2y,\ x \neq -2y$$

$$\begin{aligned}
\frac{\frac{x}{2}-y}{\frac{x}{2}+y}-\frac{\frac{x}{2}+y}{\frac{x}{2}-y} &= \frac{(x-2y)^2-(x+2y)^2}{(x+2y)(x-2y)} \\
&= \frac{(x^2-4xy+4y^2)-(x^2+4xy+4y^2)}{x^2-4y^2} \\
&= -\frac{8xy}{x^2-4y^2}
\end{aligned}$$

Expression simplification – example 12

Question

Simplify

$$\frac{x^2 - y^2}{x - y} - \frac{x + y}{x^2 - y^2} + \frac{1}{x - y}, \quad \text{where } x \neq y,\ x \neq -y.$$

$$\begin{aligned}
\frac{x^2 - y^2}{x - y} - \frac{x + y}{x^2 - y^2} + \frac{1}{x - y} &= (x + y) - \frac{x + y}{x^2 - y^2} + \frac{1}{x - y} \\
&= (x + y) - \frac{x + y}{(x - y)(x + y)} + \frac{1}{x - y} \\
&= (x + y) - \frac{1}{x - y} + \frac{1}{x - y} \\
&= x + y
\end{aligned}$$

Expression simplification – example 13

Question

Simplify

$$\frac{x^2-y^2}{x-y}-\frac{x-y}{\frac{x-y}{x+y}-\frac{x+y}{x-y}},\qquad \text{where } x\neq y,\ x\neq -y,\ xy\neq 0$$

$$\begin{aligned}
\frac{x^2-y^2}{x-y}-\frac{x-y}{\frac{x-y}{x+y}-\frac{x+y}{x-y}} &= x+y-\frac{x-y}{\frac{(x-y)^2-(x+y)^2}{(x+y)(x-y)}} = x+y-\frac{(x-y)^2(x+y)}{-4xy}\\
&= \frac{4xy(x+y)+(x-y)^2(x+y)}{4xy}\\
&= \frac{(x+y)(x^2+y^2+2xy)}{4xy} = \frac{(x+y)^3}{4xy}
\end{aligned}$$

Expression simplification – example 14

Question

Simplify

$$\frac{x^2-y^2}{x+y}+\frac{x+y}{1-\frac{x+y}{x-y}}, \quad \text{where } x \neq y,\ x \neq -y,\ y \neq 0.$$

$$\begin{aligned}
\frac{x^2-y^2}{x+y}+\frac{x+y}{1-\frac{x+y}{x-y}} &= (x-y)+\frac{x+y}{\frac{x-y-(x+y)}{x-y}} = (x-y)+\frac{x+y}{\frac{-2y}{x-y}} \\
&= x-y-\frac{(x+y)(x-y)}{2y} = \frac{2y(x-y)-(x+y)(x-y)}{2y} \\
&= \frac{(x-y)(2y-x-y)}{2y} = -\frac{(x-y)^2}{2y}
\end{aligned}$$

Expression simplification – example 15

Question

Simplify

$$\frac{x^2-y^2}{x+y}-\frac{x+y}{1+\frac{x+y}{x-y}}, \quad \text{where } x \neq y,\ x \neq -y,\ xy \neq 0$$

$$\begin{aligned}
\frac{x^2-y^2}{x+y}-\frac{x+y}{1+\frac{x+y}{x-y}} &= (x-y)-\frac{x+y}{\frac{x-y+x+y}{x-y}} = (x-y)-\frac{x+y}{\frac{2x}{x-y}} \\
&= x-y-\frac{(x+y)(x-y)}{2x} = \frac{2x(x-y)-(x+y)(x-y)}{2x} \\
&= \frac{(x-y)(2x-x-y)}{2x} = \frac{(x-y)^2}{2x}
\end{aligned}$$

Expression simplification – example 16

Question

Evaluate

$$\frac{\log_{100} 10}{\frac{1}{10}} \times \frac{\cos(30^\circ)}{\sqrt{\sqrt{16} - 1}}$$

$$\frac{\log_{100} 10}{\frac{1}{10}} \times \frac{\cos(30^\circ)}{\sqrt{\sqrt{16} - 1}} = \frac{\frac{1}{2}}{\frac{1}{10}} \times \frac{\frac{\sqrt{3}}{2}}{\sqrt{4 - 1}} = 5 \times \frac{1}{2} = \frac{5}{2}$$

Expression simplification – example 17

Question

Evaluate

$$\frac{(1+\sin(30^\circ))^2}{\left(\frac{5}{2}-\log_{10}100\right)^{-1}-1}$$

$$\begin{aligned}\frac{(1+\sin(30^\circ))^2}{\left(\frac{5}{2}-\log_{10}100\right)^{-1}-1} &= \frac{(1+\frac{1}{2})^2}{\left(\frac{5}{2}-2\right)^{-1}-1}\\ &= \frac{\left(\frac{3}{2}\right)^2}{\left(\frac{1}{2}\right)^{-1}-1} = \frac{\frac{9}{4}}{2-1} = \frac{9}{4}\end{aligned}$$

Expression simplification – example 18

Question

Evaluate

$$\frac{(1+\sin(30^\circ))^2}{\left(\frac{5}{2}-\log_{10}100\right)^{-2}-1}$$

$$\begin{aligned}
\frac{(1+\sin(30^\circ))^2}{\left(\frac{5}{2}-\log_{10}100\right)^{-2}-1} &= \frac{(1+\frac{1}{2})^2}{\left(\frac{5}{2}-2\right)^{-2}-1} \\
&= \frac{\left(\frac{3}{2}\right)^2}{\left(\frac{1}{2}\right)^{-2}-1} = \frac{\frac{9}{4}}{4-1} = \frac{3}{4}
\end{aligned}$$

Expression simplification – example 19

Question

Evaluate

$$\frac{\log_{100} 10}{\frac{1}{10}} \times \frac{\cos(60^\circ)}{\sqrt{\sqrt{25} - 1}}$$

$$\begin{aligned}
\frac{\log_{100} 10}{\frac{1}{10}} \times \frac{\cos(60^\circ)}{\sqrt{\sqrt{25} - 1}} &= \frac{\frac{1}{2}}{\frac{1}{10}} \times \frac{\frac{1}{2}}{\sqrt{5 - 1}} \\
&= 5 \times \frac{\frac{1}{2}}{\sqrt{4}} = 5 \times \frac{1}{4} = \frac{5}{4}
\end{aligned}$$

Expression simplification – example 20

Question

Evaluate

$$\frac{(\cos(60^\circ)+\sin(30^\circ))^2}{\left(\frac{5}{2}-\log_{10}100\right)^{-2}-1}$$

$$\frac{(\cos(60^\circ)+\sin(30^\circ))^2}{\left(\frac{5}{2}-2\right)^{-2}-1} = \frac{\left(\frac{1}{2}+\frac{1}{2}\right)^2}{\left(\frac{1}{2}\right)^{-2}-1} = \frac{1^2}{4-1} = \frac{1}{3}$$

3. Polynomial factorization

Polynomial factorization examples 1 – 3

$$\begin{aligned} 9p^4(a-b) - 16q^2(a-b) &= (9p^4 - 16q^2)(a-b) \\ &= (3p^2 - 4q)(3p^2 + 4q)(a-b) \end{aligned}$$

$$\begin{aligned} m^2 - n^2 - p^2 + 2np &= m^2 - (n^2 + p^2 - 2np) \\ &= m^2 - (n-p)^2 = (m+n-p)(m-n+p) \end{aligned}$$

$$\begin{aligned} (a-b)x^4 + (b-a)x^2 &= (a-b)(x^4 - x^2) = (a-b)x^2(x^2 - 1) \\ &= (a-b)x^2(x-1)(x+1) \end{aligned}$$

Polynomial factorization examples 4 – 6

$$\begin{aligned} 9(2a-x)^2 - 4(3a-x)^2 &= (3(2a-x))^2 - (2(3a-x))^2 \\ &= (6a-3x+6a-2x)(6a-3x-6a+2x) \\ &= (12a-5x)(-x) \end{aligned}$$

$$\begin{aligned} a^2y - aby + a^3y - ab^2y &= ay(a-b+a^2-b^2) = ay(a-b+(a-b)(a+b)) \\ &= ay(a-b)(1+a+b) \end{aligned}$$

$$\begin{aligned} a^6 - a^4 + 2a^3 + 2a^2 &= a^2(a^4-a^2+2a+2) = a^2(a^2(a^2-1)+2(a+1)) \\ &= a^2(a^2(a+1)(a-1)+2(a+1)) \\ &= a^2(a+1)(a^3-a^2+2) \end{aligned}$$

Polynomial factorization examples 7 – 9

$$\begin{aligned} 2a^5+6a^4+6a^3+2a^2 &= 2a^2(a^3+3a^2+3a+1) \\ &= 2a^2(a+1)^3 \end{aligned}$$

$$\begin{aligned} m^5+m^3-m^2-1 &= m^2(m^3-1)+(m^3-1) \\ &= (m^3-1)(m^2+1) = (m-1)(m^2+m+1)(m^2+1) \end{aligned}$$

$$\begin{aligned} 2x^4-x^3+x-2 &= x(1-x^2)+2(x^4-1) = x(1-x^2)+2(x^2-1)(x^2+1) \\ &= (x+1)(x-1)(-x+2x^2+2) \end{aligned}$$

Polynomial factorization examples 10 – 12

$$\begin{aligned}(4p+3q)^2-16(p-q)^2 &= ((4p+3q)-4(p-q))((4p+3q)+4(p-q)) \\ &= (4p+3q-4p+4q)(4p+3q+4p-4q) \\ &= 7q(8p-q)\end{aligned}$$

$$9p^2(a-b)-25q^2(a-b) = (a-b)(9p^2-25q^2)=(a-b)(3p-5q)(3p+5q)$$

$$25q^2(a-b)-9p^2(a-b) = (a-b)(25q^2-9p^2)=(a-b)(5q-3p)(5q+3p)$$

Polynomial factorization – example 13 – 1/2

Question

Factor the polynomial:

$$x^4 - 2x^3 + x^2 + 2x - 2$$

$$x^2(x^2 - 2x + 1) + 2(x - 1) = x^2(x - 1)^2 + 2(x - 1) = (x - 1)(x^3 - x^2 + 2)$$

To factor further, we try to find a root of $x^3 - x^2 + 2$. For this, recall the Rational Root Theorem.

Rational Root Theorem

Statement

Let

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

be a polynomial with integer coefficients. If $\frac{p}{q}$ (in simplest form) is a rational root of $f(x) = 0$, then:

- p is a factor of the constant term a_0,
- q is a factor of the leading coefficient a_n.

Implication

All possible rational roots of $f(x)$ are of the form

$$\frac{p}{q}, \quad \text{where } p \mid a_0 \text{ and } q \mid a_n$$

Polynomial factorization – example 13 – 2/2

To factor $x^3 - x^2 + 2$, we apply the Rational Root Theorem. The possible rational roots are $\pm 1, \pm 2$. Substituting, we find that:

$$x = -1 \quad \text{is a root.}$$

We divide by $x + 1$ using polynomial division:

$$\begin{array}{r|rrrr}
 & & x^2 & -2x & +2 \\ \hline
x+1 & x^3 & -x^2 & & +2 \\
 & -x^3 & -x^2 & & \\ \hline
 & & -2x^2 & & \\
 & & 2x^2 & +2x & \\ \hline
 & & & 2x & +2 \\
 & & & -2x & -2 \\ \hline
 & & & & 0
\end{array}$$

Finally

$$x^4 - 2x^3 + x^2 + 2x - 2 = (x-1)(x+1)(x^2 - 2x + 2)$$

Polynomial factorization – example 14 – 1/2

Question

Factor the polynomial:

$$x^4 - x^3 + x^2 + x - 2$$

According to the Rational Root Theorem, we test the possible rational roots: ± 1, ± 2.
Both $x = 1$ and $x = -1$ are roots of the polynomial, so $(x-1)(x+1) = x^2 - 1$ is a factor.
We proceed with polynomial division to factor out $x^2 - 1$:

$$\begin{array}{r|l} & x^2 - x + 2 \\ \hline x^2 - 1 & x^4 - x^3 + x^2 + x - 2 \\ & -x^4 \quad\quad + x^2 \\ \hline & -x^3 + 2x^2 + x \\ & x^3 \quad\quad - x \\ \hline & 2x^2 \quad\quad - 2 \\ & -2x^2 \quad\quad + 2 \\ \hline & 0 \end{array}$$

Polynomial factorization – example 14 – 2/2

Question

Factor the polynomial:

$$x^4 - x^3 + x^2 + x - 2$$

$$
\begin{array}{r}
x^2 - x + 2 \\
x^2 - 1 \overline{\big)\, x^4 - x^3 + x^2 + x - 2} \\
-x^4 \qquad\quad + x^2 \qquad\qquad \\
\hline
-x^3 + 2x^2 + x \quad\; \\
x^3 \qquad\quad - x \quad\; \\
\hline
2x^2 \qquad - 2 \\
-2x^2 \qquad + 2 \\
\hline
0
\end{array}
$$

Hence, the complete factorization is:

$$x^4 - x^3 + x^2 + x - 2 = (x + 1)(x - 1)(x^2 - x + 2)$$

Polynomial factorization —example 15

Question

Factor the polynomial:

$$x^4 + x^3 + x^2 - x - 2$$

Same as in the previous example, we find that $(x-1)(x+1) = x^2 - 1$ is a factor. We divide the polynomial by $x^2 - 1$:

$$\begin{array}{r|l}
 & x^2 + x + 2 \\
\hline
x^2 - 1 & x^4 + x^3 + x^2 - x - 2 \\
 & -x^4 \qquad\quad + x^2 \\
\hline
 & x^3 + 2x^2 - x \\
 & -x^3 \qquad\quad + x \\
\hline
 & 2x^2 \qquad - 2 \\
 & -2x^2 \qquad + 2 \\
\hline
 & 0
\end{array}$$

$$\implies x^4 + x^3 + x^2 - x - 2 = (x+1)(x-1)(x^2 + x + 2)$$

4. Solving rational equations

Solving rational equations – example 1

Question

Solve

$$\frac{x^5 - 17x^3 + 12x^2 + 52x - 48}{x^3 - x^2 - 4x + 4} = 0$$

$$\begin{array}{r}
x^2 \quad + x - 12 \\
x^3 - x^2 - 4x + 4 \overline{) \; x^5 \qquad - 17x^3 + 12x^2 + 52x - 48} \\
\underline{- x^5 + x^4 \;\; + 4x^3 \;\; - 4x^2 \qquad\qquad\quad} \\
x^4 - 13x^3 \;\; + 8x^2 + 52x \qquad \\
\underline{- x^4 \quad + x^3 \;\; + 4x^2 \;\; - 4x \qquad} \\
- 12x^3 + 12x^2 + 48x - 48 \\
\underline{12x^3 - 12x^2 - 48x + 48} \\
0
\end{array}$$

Solving rational equations – example 1

Question

Solve

$$\frac{x^5 - 17x^3 + 12x^2 + 52x - 48}{x^3 - x^2 - 4x + 4} = 0$$

$$\frac{x^5 - 17x^3 + 12x^2 + 52x - 48}{x^3 - x^2 - 4x + 4} = x^2 + x - 12$$

$$x^2 + x - 12 = 0 \Longrightarrow x = -4,\ 3$$

Solving rational equations – example 2

Question

Solve

$$\frac{x^5 - 17x^3 + 12x^2 + 52x - 48}{x^3 - 6x^2 + 11x - 6} = 0$$

$$\begin{array}{r|l}
 & x^2 + 6x + 8 \\
\hline
x^3 - 6x^2 + 11x - 6 & x^5 - 17x^3 + 12x^2 + 52x - 48 \\
 & -x^5 + 6x^4 - 11x^3 + 6x^2 \\
\hline
 & 6x^4 - 28x^3 + 18x^2 + 52x \\
 & -6x^4 + 36x^3 - 66x^2 + 36x \\
\hline
 & 8x^3 - 48x^2 + 88x - 48 \\
 & -8x^3 + 48x^2 - 88x + 48 \\
\hline
 & 0
\end{array}$$

Solving rational equations – example 2

Question

Solve

$$\frac{x^5 - 17x^3 + 12x^2 + 52x - 48}{x^3 - 6x^2 + 11x - 6} = 0$$

$$\frac{x^5 - 17x^3 + 12x^2 + 52x - 48}{x^3 - 6x^2 + 11x - 6} = x^2 + 6x + 8$$

$$x^2 + 6x + 8 = 0 \Longrightarrow x = -4, \ -2$$

Solving rational equations – example 3

Question

Solve

$$\frac{x^5 - 17x^3 + 12x^2 + 52x - 48}{x^3 - 2x^2 - 5x + 6} = 0$$

$$\begin{array}{r}
x^2 + 2x - 8 \\
x^3 - 2x^2 - 5x + 6 \overline{) \; x^5 \qquad - 17x^3 + 12x^2 + 52x - 48} \\
\underline{- x^5 + 2x^4 + 5x^3 - 6x^2 \qquad\qquad} \\
2x^4 - 12x^3 + 6x^2 + 52x \quad \\
\underline{- 2x^4 + 4x^3 + 10x^2 - 12x \quad} \\
- 8x^3 + 16x^2 + 40x - 48 \\
\underline{8x^3 - 16x^2 - 40x + 48} \\
0
\end{array}$$

Solving rational equations – example 3

Question

Solve

$$\frac{x^5 - 17x^3 + 12x^2 + 52x - 48}{x^3 - 2x^2 - 5x + 6} = 0$$

$$\frac{x^5 - 17x^3 + 12x^2 + 52x - 48}{x^3 - 2x^2 - 5x + 6} = x^2 + 2x - 8$$

$$x^2 + 2x - 8 = 0 \Longrightarrow x = -4,\ 2$$

Solving rational equations – example 4

Question

Solve

$$\frac{x^5 - 17x^3 + 12x^2 + 52x - 48}{x^3 + x^2 - 10x + 8} = 0$$

$$\begin{array}{r|rrrrrr}
 & & & x^2 & -x & -6 & \\
\hline
x^3 + x^2 - 10x + 8 & x^5 & & -17x^3 & +12x^2 & +52x & -48 \\
 & -x^5 & -x^4 & +10x^3 & -8x^2 & & \\
\hline
 & & -x^4 & -7x^3 & +4x^2 & +52x & \\
 & & x^4 & +x^3 & -10x^2 & +8x & \\
\hline
 & & & -6x^3 & -6x^2 & +60x & -48 \\
 & & & 6x^3 & +6x^2 & -60x & +48 \\
\hline
 & & & & & & 0
\end{array}$$

Solving rational equations – example 4

Question

Solve

$$\frac{x^5 - 17x^3 + 12x^2 + 52x - 48}{x^3 + x^2 - 10x + 8} = 0$$

$$\frac{x^5 - 17x^3 + 12x^2 + 52x - 48}{x^3 + x^2 - 10x + 8} = x^2 - x - 6$$

$$x^2 - x - 6 = 0 \implies x = 3, \ -2$$

Solving rational equations – example 5

Question

Solve

$$\frac{x^5 - 17x^3 + 12x^2 + 52x - 48}{x^3 + 5x^2 + 2x - 8} = 0$$

$$\begin{array}{r|l}
 & x^2 - 5x + 6 \\ \hline
x^3 + 5x^2 + 2x - 8 & x^5 \qquad - 17x^3 + 12x^2 + 52x - 48 \\
 & -x^5 - 5x^4 - 2x^3 + 8x^2 \\ \cline{2-2}
 & -5x^4 - 19x^3 + 20x^2 + 52x \\
 & 5x^4 + 25x^3 + 10x^2 - 40x \\ \cline{2-2}
 & 6x^3 + 30x^2 + 12x - 48 \\
 & -6x^3 - 30x^2 - 12x + 48 \\ \cline{2-2}
 & 0
\end{array}$$

Solving rational equations – example 5

Question

Solve

$$\frac{x^5 - 17x^3 + 12x^2 + 52x - 48}{x^3 + 5x^2 + 2x - 8} = 0$$

$$\frac{x^5 - 17x^3 + 12x^2 + 52x - 48}{x^3 + 5x^2 + 2x - 8} = x^2 - 5x + 6$$

$$x^2 - 5x + 6 = 0 \implies x = 3,\ 2$$

Solving rational equations – example 6

Question

Solve

$$\frac{2x^5 - 3x^4 - 28x^3 + 63x^2 - 10x - 24}{x^3 - 6x^2 + 11x - 6} = 0$$

$$\begin{array}{r|l}
 & 2x^2 \quad + 9x \quad + 4 \\
\hline
x^3 - 6x^2 + 11x - 6 & 2x^5 \quad - 3x^4 - 28x^3 + 63x^2 - 10x - 24 \\
 & -2x^5 + 12x^4 - 22x^3 + 12x^2 \\
\hline
 & 9x^4 - 50x^3 + 75x^2 - 10x \\
 & -9x^4 + 54x^3 - 99x^2 + 54x \\
\hline
 & 4x^3 - 24x^2 + 44x - 24 \\
 & -4x^3 + 24x^2 - 44x + 24 \\
\hline
 & 0
\end{array}$$

Solving rational equations – example 6

Question

Solve

$$\frac{2x^5 - 3x^4 - 28x^3 + 63x^2 - 10x - 24}{x^3 - 6x^2 + 11x - 6} = 0$$

$$\frac{2x^5 - 3x^4 - 28x^3 + 63x^2 - 10x - 24}{x^3 - 6x^2 + 11x - 6} = 2x^2 + 9x + 4$$

$$2x^2 + 9x + 4 = 0 \Longrightarrow x = -4,\ -\frac{1}{2}$$

Solving rational equations – example 7

Question

Solve

$$\frac{2x^5 - 3x^4 - 28x^3 + 63x^2 - 10x - 24}{x^3 - x^2 - 14x + 24} = 0$$

$$
\begin{array}{r|rrrrrr}
 & & & & 2x^2 & -x & -1 \\ \hline
x^3 - x^2 - 14x + 24 & 2x^5 & -3x^4 & -28x^3 & +63x^2 & -10x & -24 \\
 & -2x^5 & +2x^4 & +28x^3 & -48x^2 & & \\ \hline
 & & -x^4 & & +15x^2 & -10x & \\
 & & x^4 & -x^3 & -14x^2 & +24x & \\ \hline
 & & & -x^3 & +x^2 & +14x & -24 \\
 & & & x^3 & -x^2 & -14x & +24 \\ \hline
 & & & & & & 0
\end{array}
$$

Solving rational equations – example 7

Question

Solve

$$\frac{2x^5 - 3x^4 - 28x^3 + 63x^2 - 10x - 24}{x^3 - x^2 - 14x + 24} = 0$$

$$\frac{2x^5 - 3x^4 - 28x^3 + 63x^2 - 10x - 24}{x^3 - x^2 - 14x + 24} = 2x^2 - x - 1$$

$$2x^2 - x - 1 = 0 \Longrightarrow x = 1,\ -\frac{1}{2}$$

Solving rational equations – example 8

Question

Solve

$$\frac{2x^5 - 3x^4 - 28x^3 + 63x^2 - 10x - 24}{x^3 + x^2 - 10x + 8} = 0$$

$$\begin{array}{r|l}
 & 2x^2 \quad - 5x \quad - 3 \\
\hline
x^3 + x^2 - 10x + 8 & 2x^5 - 3x^4 - 28x^3 + 63x^2 - 10x - 24 \\
 & -2x^5 - 2x^4 + 20x^3 - 16x^2 \\
 & \quad - 5x^4 - 8x^3 + 47x^2 - 10x \\
 & \quad 5x^4 + 5x^3 - 50x^2 + 40x \\
 & \qquad - 3x^3 - 3x^2 + 30x - 24 \\
 & \qquad 3x^3 + 3x^2 - 30x + 24 \\
 & \qquad\qquad 0
\end{array}$$

Solving rational equations – example 8

Question

Solve

$$\frac{2x^5 - 3x^4 - 28x^3 + 63x^2 - 10x - 24}{x^3 + x^2 - 10x + 8} = 0$$

$$\frac{2x^5 - 3x^4 - 28x^3 + 63x^2 - 10x - 24}{x^3 + x^2 - 10x + 8} = 2x^2 - 5x - 3$$

$$2x^2 - 5x - 3 = 0 \Longrightarrow x = 3,\ -\frac{1}{2}$$

Solving rational equations – example 9

Question

Solve

$$\frac{2x^5 - 3x^4 - 28x^3 + 63x^2 - 10x - 24}{2x^3 - 5x^2 + x + 2} = 0$$

$$
\begin{array}{r|rrrrrr}
 & & & & x^2 & +x & -12 \\
\hline
2x^3 - 5x^2 + x + 2) & 2x^5 & -3x^4 & -28x^3 & +63x^2 & -10x & -24 \\
 & -2x^5 & +5x^4 & -x^3 & -2x^2 & & \\
\hline
 & & 2x^4 & -29x^3 & +61x^2 & -10x & \\
 & & -2x^4 & +5x^3 & -x^2 & -2x & \\
\hline
 & & & -24x^3 & +60x^2 & -12x & -24 \\
 & & & 24x^3 & -60x^2 & +12x & +24 \\
\hline
 & & & & & & 0
\end{array}
$$

Solving rational equations – example 9

Question

Solve

$$\frac{2x^5 - 3x^4 - 28x^3 + 63x^2 - 10x - 24}{2x^3 - 5x^2 + x + 2} = 0$$

$$\frac{2x^5 - 3x^4 - 28x^3 + 63x^2 - 10x - 24}{2x^3 - 5x^2 + x + 2} = x^2 + x - 12$$

$$x^2 + x - 12 = 0 \Longrightarrow x = -4,\ 3$$

Solving rational equations – example 10

Question

Solve

$$\frac{2x^5 - 3x^4 - 44x^3 + 103x^2 - 18x - 40}{2x^3 - 5x^2 + x + 2} = 0$$

$$\begin{array}{r|l}
 & x^2 + x - 20 \\
\hline
2x^3 - 5x^2 + x + 2 & 2x^5 - 3x^4 - 44x^3 + 103x^2 - 18x - 40 \\
 & -2x^5 + 5x^4 - x^3 - 2x^2 \\
\hline
 & 2x^4 - 45x^3 + 101x^2 - 18x \\
 & -2x^4 + 5x^3 - x^2 - 2x \\
\hline
 & -40x^3 + 100x^2 - 20x - 40 \\
 & 40x^3 - 100x^2 + 20x + 40 \\
\hline
 & 0
\end{array}$$

Solving rational equations – example 10

Question

Solve

$$\frac{2x^5 - 3x^4 - 44x^3 + 103x^2 - 18x - 40}{2x^3 - 5x^2 + x + 2} = 0$$

$$\frac{2x^5 - 3x^4 - 44x^3 + 103x^2 - 18x - 40}{2x^3 - 5x^2 + x + 2} = x^2 + x - 20$$

$$x^2 + x - 20 = 0 \Longrightarrow x = -5,\ 4$$

Solving rational equations – example 11

Question

Solve

$$\frac{2x^5 - 7x^4 - 18x^3 + 61x^2 - 14x - 24}{2x^3 - 5x^2 + x + 2} = 0$$

$$
\begin{array}{r|rrrrrr}
 & & & & x^2 & -\,x & -\,12 \\
\cline{2-7}
2x^3 - 5x^2 + x + 2 & 2x^5 & -\,7x^4 & -\,18x^3 & +\,61x^2 & -\,14x & -\,24 \\
 & -\,2x^5 & +\,5x^4 & -\,x^3 & -\,2x^2 & & \\
\cline{2-5}
 & & -\,2x^4 & -\,19x^3 & +\,59x^2 & -\,14x & \\
 & & 2x^4 & -\,5x^3 & +\,x^2 & +\,2x & \\
\cline{3-6}
 & & & -\,24x^3 & +\,60x^2 & -\,12x & -\,24 \\
 & & & 24x^3 & -\,60x^2 & +\,12x & +\,24 \\
\cline{4-7}
 & & & & & & 0
\end{array}
$$

Solving rational equations – example 11

Question

Solve

$$\frac{2x^5 - 7x^4 - 18x^3 + 61x^2 - 14x - 24}{2x^3 - 5x^2 + x + 2} = 0$$

$$\frac{2x^5 - 7x^4 - 18x^3 + 61x^2 - 14x - 24}{2x^3 - 5x^2 + x + 2} = x^2 - x - 12$$

$$x^2 - x - 12 = 0 \Longrightarrow x = -3,\ 4$$

Solving rational equations – example 12

Question

Solve

$$\frac{x^5 - x^4 - 19x^3 + 25x^2 + 66x - 72}{x^3 - 13x + 12} = 0$$

$$\begin{array}{r|rrrrrr}
 & & & & x^2 & -x & -6 \\ \hline
x^3 - 13x + 12 & x^5 & -x^4 & -19x^3 & +25x^2 & +66x & -72 \\
 & -x^5 & & +13x^3 & -12x^2 & & \\ \hline
 & & -x^4 & -6x^3 & +13x^2 & +66x & \\
 & & x^4 & & -13x^2 & +12x & \\ \hline
 & & & -6x^3 & & +78x & -72 \\
 & & & 6x^3 & & -78x & +72 \\ \hline
 & & & & & & 0
\end{array}$$

Solving rational equations – example 12

Question

Solve

$$\frac{x^5 - x^4 - 19x^3 + 25x^2 + 66x - 72}{x^3 - 13x + 12} = 0$$

$$\frac{x^5 - x^4 - 19x^3 + 25x^2 + 66x - 72}{x^3 - 13x + 12} = x^2 - x - 6$$

$$x^2 - x - 6 = 0 \implies x = -2,\ 3$$

Solving rational equations – example 13

Question

Solve

$$\frac{2x^5 - 3x^4 - 16x^3 + 33x^2 - 4x - 12}{2x^3 - 5x^2 + x + 2} = 0$$

$$\begin{array}{r|l}
 & x^2 + x - 6 \\
\hline
2x^3 - 5x^2 + x + 2 & 2x^5 - 3x^4 - 16x^3 + 33x^2 - 4x - 12 \\
 & -2x^5 + 5x^4 - x^3 - 2x^2 \\
\hline
 & 2x^4 - 17x^3 + 31x^2 - 4x \\
 & -2x^4 + 5x^3 - x^2 - 2x \\
\hline
 & -12x^3 + 30x^2 - 6x - 12 \\
 & 12x^3 - 30x^2 + 6x + 12 \\
\hline
 & 0
\end{array}$$

Solving rational equations – example 13

Question

Solve

$$\frac{2x^5 - 3x^4 - 16x^3 + 33x^2 - 4x - 12}{2x^3 - 5x^2 + x + 2} = 0$$

$$\frac{2x^5 - 3x^4 - 16x^3 + 33x^2 - 4x - 12}{2x^3 - 5x^2 + x + 2} = x^2 + x - 6$$

$$x^2 + x - 6 = 0 \Longrightarrow x = -3,\ 2$$

Solving rational equations – example 14

Question

Solve

$$\frac{2x^5 + x^4 - 22x^3 + 25x^2 + 2x - 8}{2x^3 - 5x^2 + x + 2} = 0$$

$$\begin{array}{r}
x^2 + 3x - 4 \\
2x^3 - 5x^2 + x + 2 \overline{) \; 2x^5 + x^4 - 22x^3 + 25x^2 + 2x - 8} \\
\underline{-2x^5 + 5x^4 - x^3 - 2x^2 \qquad\qquad} \\
6x^4 - 23x^3 + 23x^2 + 2x \quad \\
\underline{-6x^4 + 15x^3 - 3x^2 - 6x \quad} \\
-8x^3 + 20x^2 - 4x - 8 \\
\underline{8x^3 - 20x^2 + 4x + 8} \\
0
\end{array}$$

Solving rational equations – example 14

Question

Solve

$$\frac{2x^5 + x^4 - 22x^3 + 25x^2 + 2x - 8}{2x^3 - 5x^2 + x + 2} = 0$$

$$\frac{2x^5 + x^4 - 22x^3 + 25x^2 + 2x - 8}{2x^3 - 5x^2 + x + 2} = x^2 + 3x - 4$$

$$x^2 + 3x - 4 = 0 \Longrightarrow x = -4,\ 1$$

Solving rational equations – example 15

Question

Solve

$$\frac{2x^5 + 7x^4 - 11x^3 - 37x^2 + 21x + 18}{2x^3 - 5x^2 + x + 2} = 0$$

$$
\begin{array}{r|llllll}
 & & & & x^2 & +6x & +9 \\
\hline
2x^3 - 5x^2 + x + 2 & 2x^5 & +7x^4 & -11x^3 & -37x^2 & +21x & +18 \\
 & -2x^5 & +5x^4 & -x^3 & -2x^2 & & \\
\cline{2-5}
 & & 12x^4 & -12x^3 & -39x^2 & +21x & \\
 & & -12x^4 & +30x^3 & -6x^2 & -12x & \\
\cline{3-6}
 & & & 18x^3 & -45x^2 & +9x & +18 \\
 & & & -18x^3 & +45x^2 & -9x & -18 \\
\cline{4-7}
 & & & & & & 0
\end{array}
$$

Solving rational equations – example 15

Question

Solve

$$\frac{2x^5 + 7x^4 - 11x^3 - 37x^2 + 21x + 18}{2x^3 - 5x^2 + x + 2} = 0$$

$$\frac{2x^5 + 7x^4 - 11x^3 - 37x^2 + 21x + 18}{2x^3 - 5x^2 + x + 2} = x^2 + 6x + 9$$

$$x^2 + 6x + 9 = 0 \Longrightarrow x = -3$$

Solving rational equations – example 16

Question

Solve

$$\frac{2x^5 - 15x^4 + 38x^3 - 33x^2 - 4x + 12}{2x^3 - 5x^2 + x + 2} = 0$$

$$\begin{array}{r|l}
 & x^2 - 5x + 6 \\
\hline
2x^3 - 5x^2 + x + 2) & 2x^5 - 15x^4 + 38x^3 - 33x^2 - 4x + 12 \\
 & -2x^5 + 5x^4 - x^3 - 2x^2 \\
\hline
 & -10x^4 + 37x^3 - 35x^2 - 4x \\
 & 10x^4 - 25x^3 + 5x^2 + 10x \\
\hline
 & 12x^3 - 30x^2 + 6x + 12 \\
 & -12x^3 + 30x^2 - 6x - 12 \\
\hline
 & 0
\end{array}$$

Solving rational equations – example 16

Question

Solve

$$\frac{2x^5 - 15x^4 + 38x^3 - 33x^2 - 4x + 12}{2x^3 - 5x^2 + x + 2} = 0$$

$$\frac{2x^5 - 15x^4 + 38x^3 - 33x^2 - 4x + 12}{2x^3 - 5x^2 + x + 2} = x^2 - 5x + 6$$

$$x^2 - 5x + 6 = 0 \Longrightarrow x = 2,\ 3$$

5. Real roots and solution intervals

Real roots and solution intervals – example 1

Question

Solve

$$x^2 - x - 3 = \frac{2-x}{x-2}$$

$$x^2 - x - 3 = \frac{2-x}{x-2} \implies x^2 - x - 2 = 0, \;\; x \neq 2$$

We have

$$x = -1$$

Real roots and solution intervals – example 2

Question

Solve

$$x^4 - 3x^2 + 3 = \frac{x-1}{x-1}$$

$$x^4 - 3x^2 + 3 = \frac{x-1}{x-1} \Longrightarrow x^4 - 3x^2 + 2 = 0, \;\; x \neq 1$$

Let $y = x^2$, we get

$$y^2 - 3y + 2 = 0 \Longrightarrow y = 1, 2 \Longrightarrow x = \pm 1, \pm\sqrt{2}$$

We have

$$x = \pm\sqrt{2}, -1$$

Real roots and solution intervals – example 3

Question

Find the set of all real numbers that satisfy the following inequality

$$2x^2 + x + 2 \leq 7 - 2x$$

$$2x^2 + x + 2 \leq 7 - 2x \implies 2x^2 + 3x - 5 \leq 0$$

$$2x^2 + 3x - 5 = 0 \implies x = -\frac{5}{2}, 1$$

Since the leading coefficient of the quadratic function $f(x) = 2x^2 + 3x - 5$ is positive, the graph of f is concave upward. Therefore, the solution set to the inequality corresponds to the values of x for which the quadratic expression lies on or below the x-axis

$$\left[-\frac{5}{2},\ 1\right]$$

Real roots and solution intervals – example 4

Question

Find the set of all real numbers that satisfy the following inequality

$$x^4 - 3x^2 + 7 \geq x^2 - 1$$

$$x^4 - 3x^2 + 7 \geq x^2 - 1 \implies x^4 - 4x^2 + 8 \geq 0$$

Note that

$$x^4 - 4x^2 + 8 = (x^2 - 2)^2 + 4 \geq 4$$

Thus all $x \in \mathbb{R}$ satisfies the inequality

Real roots and solution intervals – example 5

Question

Find the set of all real numbers that satisfy the following inequality

$$x^4 + 7x^2 - 8 \leq 0$$

Let $y = x^2$,

$$y^2 + 7y - 8 = 0 \implies y = -8, 1$$

$$y^2 + 7y - 8 \leq 0 \implies -8 \leq y \leq 1$$

Since $y = x^2 \geq 0$, we get

$$x^4 + 7x^2 - 8 \leq 0 \implies x^2 \leq 1 \implies -1 \leq x \leq 1$$

Real roots and solution intervals – example 6

Question

Find the set of all real numbers that satisfy the following inequality

$$x^2 - 3x - 5 \leq \frac{4-x}{x-4}$$

$$x^2 - 3x - 5 \leq \frac{4-x}{x-4} \Longrightarrow x^2 - 3x - 4 \leq 0, \;\; x \neq 4$$

$$x^2 - 3x - 4 = 0 \Longrightarrow x = -1, 4$$

Since the leading coefficient of the quadratic function is positive, the graph is concave upward. The solution set is given by

$$[-1, 4)$$

Real roots and solution intervals – example 7

Question

Find the set of all real numbers that satisfy the following inequality

$$\frac{x}{x-2} - \frac{3}{x+1} \leq 1$$

We have $x \neq -1, 2$ and

$$\frac{x(x+1)}{(x-2)(x+1)} - \frac{3(x-2)}{(x-2)(x+1)} - 1 \leq 0 \Longrightarrow \frac{x^2 + x - 3x + 6 - x^2 - x + 2x + 2}{(x-2)(x+1)} \leq 0$$

$$\Longrightarrow \frac{8-x}{(x-2)(x+1)} \leq 0$$

Thus $x \leq 8, (x-2)(x+1) \leq 0$ or $x \geq 8, (x-2)(x+1) \geq 0$, which implies

$$x \leq 8, x \leq 2, x \geq -1, \text{ or } x \leq 8, x \geq 2, x \leq -1, \text{ or } x \geq 8$$

Consequently, $x \in (-1, 2) \cup [8, \infty)$

Real roots and solution intervals – example 8

Question

Find the set of all real numbers that satisfy the following inequality

$$\sqrt{2x-1} \leq x-2$$

We have $x \geq \dfrac{1}{2}$ and

$$2x-1 \leq x^2+4-4x \Longrightarrow x^2-6x+5 \geq 0 \Longrightarrow x \geq 5 \text{ or } x \leq 1$$

Thus, $x \geq 5$

Real roots and solution intervals – example 9

Question

The quadratic equation

$$x^2 - 9x + q = 0$$

has one root -42. Find the second root and the value of q.

Let $x_1 = -42$ and x_2 be the two roots of the equation. By Vieta's formulas

$$x_1 + x_2 = 9, \quad x_1 x_2 = q \implies$$

$$x_2 = 9 - (-42) = 51, \quad q = (-42) \times 51 = -2142$$

Real roots and solution intervals – example 10

Question

Find the set of all real numbers that satisfy the following equation

$$x^2 - 3x + 1 = \frac{2 - x}{x - 2}$$

We begin by identifying the domain: the denominator $x - 2 \neq 0$, so $x \neq 2$.

$$x^2 - 3x + 1 = -1 \Longrightarrow x = 1,\ 2$$

In conclusion $x = 1$

Real roots and solution intervals – example 11

Question

Solve

$$x^4 - 3x^2 + 1 = \frac{x-1}{1-x},$$

where $x \in \mathbb{R}$

Note that

$$\frac{x-1}{1-x} = -1 \quad \text{for } x \neq 1$$

Let $y = x^2$

$$x^4 - 3x^2 + 1 = -1 \Longrightarrow y^2 - 3y + 2 = 0 \Longrightarrow y = 1,\ 2 \Longrightarrow x = \pm 1,\ \pm\sqrt{2}$$

In conclusion

$$x = -1,\ \pm\sqrt{2}$$

Real roots and solution intervals – example 12

Question

Solve

$$\frac{x^2 - 3x + 1}{x - 2} = \frac{1}{2 - x},$$

where $x \in \mathbb{R}$

First note that $x \neq 2$

$$x^2 - 3x + 1 = -1 \implies x = 1,\ 2$$

In conclusion, $x = 1$

Real roots and solution intervals – example 13

Question

Solve

$$\frac{x^4 - 4x^2 + 3}{x^2 - 1} = 1,$$

where $x \in \mathbb{R}$.

First note that $x \neq \pm 1$. Let $y = x^2$

$$y^2 - 4y + 3 = y - 1 \implies y = 1,\ 4 \implies x = \pm 2$$

Real roots and solution intervals – example 14

Question

Solve

$$x^4 - 8x^2 - 9 \leq 0,$$

where $x \in \mathbb{R}$.

Let $y = x^2$

$$y^2 - 8y - 9 \leq 0 \implies -1 \leq y \leq 9 \implies 0 \leq y \leq 9 \implies -3 \leq x \leq 3$$

Real roots and solution intervals – example 15

Question

Solve

$$x^2 - 3x - 5 = \frac{4-x}{x-4},$$

where $x \in \mathbb{R}$.

First, note that the equation is undefined for $x = 4$

$$x^2 - 3x - 5 = -1 \Longrightarrow x = -1,\ 4$$

Thus

$$x = -1$$

Real roots and solution intervals – example 16

Question

Solve

$$x^4 - 5x^2 + 3 = \frac{2-x}{x-2},$$

where $x \in \mathbb{R}$.

First note that $x \neq 2$. Let $y = x^2$

$$y^2 - 5y + 3 = -1 \Longrightarrow y = 1,\ 4 \Longrightarrow x = \pm 1,\ \pm 2$$

In conclusion

$$x = -2,\ \pm 1$$

Real roots and solution intervals – example 18

Question

Solve

$$x^4 + 3x^2 - 4 \leq x^2 - 1,$$

where $x \in \mathbb{R}$.

Let $y = x^2$

$$y^2 + 3y - 4 \leq y - 1 \implies -3 \leq y \leq 1 \implies 0 \leq y \leq 1 \implies -1 \leq x \leq 1$$

Real roots and solution intervals – example 19

Question

Solve

$$x^4 + 3x^2 - 4 \leq x^2 - 5,$$

where $x \in \mathbb{R}$.

Let $y = x^2$

$$y^2 + 3y - 4 \leq y - 5 \Longrightarrow y = -1$$

Since $y \geq 0$, the inequality has no solution.

6. Integer solutions

Integer solutions – example 1

Question

Find all integers that satisfy the following inequality:

$$6x^2 - 7x + 4 \leq 3$$

$$6x^2 - 7x + 4 \leq 3 \Longrightarrow 6x^2 - 7x + 1 \leq 0$$

$$6x^2 - 7x + 1 = 0 \Longrightarrow x = \frac{1}{6}, 1$$

Since the leading coefficient of the quadratic function is positive, the graph is concave upward. Therefore, the solution set to the inequality corresponds to the values of x for which the quadratic expression lies on or below the x-axis

$$\left[\frac{1}{6},\ 1\right]$$

The only integer solution is 1

Integer solutions – example 2

Question

Find all integers that satisfy the following inequality:

$$2x^2 + x - 6 \leq x^2$$

$$2x^2 + x - 6 \leq x^2 \implies x^2 + x - 6 \leq 0$$

$$x^2 + x - 6 = 0 \implies x = -3, 2$$

We have $x \in [-3, 2]$ and hence

$$x = -3,\ -2,\ -1,\ 0,\ 1,\ 2$$

Integer solutions – example 3

Question

Find all prime numbers that satisfy the following inequality:

$$\frac{7x-1}{3}+6 \geq 5x-\frac{5+3x}{2}$$

$$\frac{7x-1}{3}+6 \geq 5x-\frac{5+3x}{2} \Longrightarrow 14x-2+36-30x+15+9x \geq 0 \Longrightarrow$$
$$-7x+49 \geq 0 \Longrightarrow x \leq 7$$

We have

$$x = 2,\ 3,\ 5,\ 7$$

Integer solutions – example 4

Question

Find all integers that satisfy the following inequality:

$$\frac{3-2x}{x-7} \geq 2$$

$$\frac{3-2x}{x-7} \geq 2 \Longrightarrow \frac{3-2x-2x+14}{x-7} \geq 0, x \neq 7 \Longrightarrow \frac{17-4x}{x-7} \geq 0, x \neq 7$$

We have

$$x \leq \frac{17}{4}, x > 7, \text{ or } x \geq \frac{17}{4}, x < 7$$

Consequently

$$x = 5,\ 6$$

Integer solutions – example 5

Question

Find all integers that satisfy the following inequality:

$$\frac{3-x}{x-1} - 1 \leq \frac{3x+1}{1-x}$$

$$\frac{3-x}{x-1} - 1 \leq \frac{3x+1}{1-x} \implies \frac{3-x-x+1+3x+1}{x-1} \leq 0, x \neq 1 \implies \frac{5+x}{x-1} \leq 0, x \neq 1$$

We have

$$x \geq -5, x < 1, \text{ or } x \leq -5, x > 1$$

Consequently

$$x = -5,\ -4,\ -3,\ -2,\ -1,\ 0$$

Integer solutions – example 6

Question

Find all integers that satisfy the following inequality:

$$\frac{3x+1}{2x+1} - 1 \geq 2$$

$$\frac{3x+1}{2x+1} - 1 \geq 2 \implies \frac{3x+1-2x-1-4x-2}{2x+1} \implies \frac{-3x-2}{2x+1} \leq 0 \implies -\frac{2}{3} \leq x < -\frac{1}{2}$$

There are no integers in this interval

Integer solutions – example 7

Question

Find all positive integers that satisfy the following inequality:

$$\frac{4x+19}{x+5} < \frac{4x-17}{x-3}$$

$$\frac{4x+19}{x+5} < \frac{4x-17}{x-3} \implies \frac{(4x+19)(x-3)-(4x-17)(x+5)}{(x+5)(x-3)} < 0 \implies \frac{4x+28}{(x+5)(x-3)}$$

$$\implies x < -7 \text{ or } -5 < x < 3$$

Thus

$$x = 1,\ 2$$

Integer solutions – example 8

Question

Find all integers that satisfy the following inequality:

$$x^2 - 5x \leq 12 - x^2$$

$$x^2 - 5x - 12 + x^2 \leq 0 \Longrightarrow 2x^2 - 5x - 12 \leq 0 \Longrightarrow -\frac{3}{2} \leq x \leq 4$$

Thus

$$x = -1,\ 0,\ 1,\ 2,\ 3,\ 4$$

Integer solutions – example 9

Question

Find all integers that satisfy the following inequality:

$$2x^2 + x - 24 \leq -x$$

$$2x^2 + 2x - 24 \leq 0 \Longrightarrow -4 \leq x \leq 3$$

Thus

$$x = -4,\ -3,\ -2,\ -1,\ 0,\ 1,\ 2,\ 3$$

Integer solutions – example 10

Question

Find all integers that satisfy the following inequality:

$$-x^2 - x + 5 \geq -1$$

$$-x^2 - x + 6 \geq 0 \Longrightarrow -3 \leq x \leq 2$$

Thus

$$x = -3,\ -2,\ -1,\ 0,\ 1,\ 2$$

Integer solutions – example 11

Question

Find all positive integers that satisfy the following inequality:

$$-x + 5 \geq 2x - 3$$

$$-3x + 8 \geq 0 \Longrightarrow x \leq \frac{8}{3}$$

Thus

$$x = 1,\ 2$$

Integer solutions – example 12

Question

Find all positive integers that satisfy the following inequality:

$$-x + 6 \geq x - 3$$

$$-2x + 9 \geq 0 \Longrightarrow x \leq \frac{9}{2}$$

Thus

$$x = 1,\ 2,\ 3,\ 4$$

Integer solutions – example 13

Question

Find all positive integers that satisfy the following inequality:

$$-x + 6 \geq x - 14$$

$$-2x + 20 \geq 0 \implies x \leq 10$$

Thus

$$x = 1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7,\ 8,\ 9,\ 10$$

Integer solutions – example 14

Question

Find all integers that satisfy the following inequality:

$$\frac{3x+1}{2x+1} - 1 \geq 1$$

$$\frac{3x+1-(2x+1)}{2x+1} \geq 1 \implies \frac{x}{2x+1} \geq 1$$

If $2x+1>0$,

$$x \geq 2x+1 \implies x \leq -1 \implies -\frac{1}{2} < x \leq -1$$

If $2x+1<0$,

$$x \leq 2x+1 \implies x \geq -1$$

In conclusion

$$-1 \leq x \leq -\frac{1}{2} \implies x = -1$$

Integer solutions – example 15

Question

Find all positive integers that satisfy the following inequality:

$$3x^2 - 5x \leq x^2 - 2$$

$$2x^2 - 5x + 2 \leq 0 \Longrightarrow \frac{1}{2} \leq x \leq 2$$

$$x = 1, 2$$

Integer solutions – example 16

Question

Find all integers that satisfy the following inequality:

$$5x^2 - 6x + 4 \leq 3$$

$$5x^2 - 6x + 1 \leq 0 \Longrightarrow \frac{1}{5} \leq x \leq 1$$

Thus

$$x = 1$$

Integer solutions – example 17

Question

Find all integers that satisfy the following inequality:

$$x^4 - 3x^2 + 5 < 2x^2 + 1,$$

Let $y = x^2$

$$y^2 - 3y + 5 < 2y + 1 \implies 1 < y < 4 \implies x \in (-2, -1) \cup (1, 2)$$

The inequality does not have any integer solutions.

Integer solutions – example 18

Question

Find all integers that satisfy the following inequality:

$$x^4 - 3x^2 + 5 \leq 2x^2 + 1,$$

Let $y = x^2$

$$y^2 - 3y + 5 \leq 2y + 1 \implies 1 \leq y \leq 4 \implies x \in [-2, -1] \cup [1, 2]$$

Thus

$$x = -2,\ -1,\ 1,\ 2$$

7. Equations and inequalities with absolute values

Equations and inequalities with absolute values – example 1

Question

Find all integers that satisfy the following inequality:

$$|3x+1|-|x-2|<7$$

If $x \geq 2$,

$$3x+1-x+2-7<0 \Longrightarrow x<2.$$

If $-\frac{1}{3}<x<2$,

$$3x+1+x-2-7<0 \Longrightarrow x<2$$

If $x \leq -\frac{1}{3}$,

$$-3x-1+x-2-7<0 \Longrightarrow x>-5$$

In conclusion

$$-5<x<2 \Longrightarrow x=-4,\ -3,\ -2,\ -1,\ 0,\ 1$$

Equations and inequalities with absolute values – example 2

Question

Find all integers that satisfy the following equation:

$$|x-5|+|2-x|=3$$

If $x \geq 5$,

$$x-5+x-2=3 \Longrightarrow 2x=10 \Longrightarrow x=5$$

If $2 < x < 5$,

$$5-x+x-2=3 \Longrightarrow 3=3 \Longrightarrow x=3,4$$

If $x \leq 2$,

$$5-x+2-x=3 \Longrightarrow x=2$$

In conclusion,

$$x=2,\ 3,\ 4,\ 5$$

Equations and inequalities with absolute values – example 3

Question

Find all positive integers that satisfy the following inequality:

$$|x+1| + |x-4| > 7$$

If $x \geq 4$,

$$x + 1 + x - 4 > 7 \implies x > 5$$

If $-1 < x < 4$,

$$x + 1 + 4 - x > 7 \implies \text{ no solution}$$

If $x \leq -1$,

$$-x - 1 + 4 - x > 7 \implies x < -2$$

In conclusion,

$$x > 5 \text{ or } x < -2 \implies x > 5$$

Equations and inequalities with absolute values – example 4

Question

Find the set of all real numbers that satisfy the following inequality:

$$|x^2 - 2x| < x$$

If $x^2 \geq 2x$, i.e. $x \leq 0$ or $x \geq 2$

$$x^2 - 3x < 0 \Longrightarrow 0 < x < 3 \Longrightarrow 2 \leq x < 3$$

If $x^2 < 2x$, i.e. $0 < x < 2$

$$x - x^2 < 0 \Longrightarrow x < 0 \text{ or } x > 1 \Longrightarrow 1 < x < 2$$

In conclusion,

$$1 < x < 3$$

Equations and inequalities with absolute values – example 5

Question

Find the set of all real numbers that satisfy the following equation:

$$|4 - 2x| + |3 + x| = 5$$

If $4 - 2x \geq 0$, $3 + x \geq 0$ i.e. $-3 \leq x \leq 2$

$$4 - 2x + 3 + x = 5 \Longrightarrow x = 2$$

If $4 - 2x \geq 0$, $3 + x < 0$ i.e. $x < -3$

$$4 - 2x - 3 - x = 5 \Longrightarrow x = -\frac{4}{3}$$

If $x > 2$

$$2x - 4 + 3 + x = 5 \Longrightarrow x = 2$$

In conclusion,

$$x = 2$$

Equations and inequalities with absolute values – example 6

Question

Find the set of all integers that satisfy the following equation:

$$|x^2 - 2x + 2| = 5$$

Since

$$x^2 - 2x + 2 = (x-1)^2 + 1 \geq 1,$$

we have

$$x^2 - 2x + 2 = 5 \Longrightarrow x = -1,\ 3$$

Equations and inequalities with absolute values – example 7

Question

Find the set of all positive integers that satisfy the following inequality:

$$\frac{|4x-3|}{x-5} < 4$$

If $x < \frac{3}{4}$,

$$\frac{3-4x}{x-5} < 4 \Longrightarrow x < \frac{23}{8} \text{ or } x > 5$$

If $x \geq \frac{3}{4}$,

$$\frac{4x-3}{x-5} < 4 \Longrightarrow x < 5$$

Thus

$$x < 5 \Longrightarrow x = 1,\ 2,\ 3,\ 4$$

Equations and inequalities with absolute values – example 8

Question

Find the set of all real numbers that satisfy the following inequality:

$$|x+3| < |x-2|$$

If $x > 2$,

$$x+3 < x-2 \implies 3 < -2$$

If $-3 \leq x \leq 2$,

$$x+3 < 2-x \implies x < -\frac{1}{2}$$

If $x < -3$

$$-x-3 < 2-x \implies -3 < 2$$

Thus

$$x < -\frac{1}{2}$$

Equations and inequalities with absolute values – example 9

Question

Find the set of all real numbers that satisfy the following inequality:

$$|x - 2| < 2 - |x - 3|$$

If $x < 2$,

$$2 - x < 2 - (3 - x) \implies 2 - x < -1 + x \implies x > \frac{3}{2}$$

If $2 \leq x < 3$,

$$x - 2 < 2 - (3 - x) \implies x - 2 < -1 + x \implies \text{always true}$$

If $x \geq 3$,

$$x - 2 < 2 - (x - 3) \implies x - 2 < 5 - x \implies x < \frac{7}{2}$$

Thus,

$$x \in \left(\frac{3}{2}, \frac{7}{2}\right)$$

Equations and inequalities with absolute values – example 10

Question

Find the set of all real numbers that satisfy the inequality:

$$|x-2| < 2 - |x-1|$$

If $x < 1$,

$$2 - x < 2 - (1 - x) \implies 2 - x < 1 + x \implies x > \frac{1}{2}$$

If $1 \leq x < 2$,

$$2 - x < 2 - (x - 1) \implies 2 - x < 3 - x \implies \text{always true}$$

If $x \geq 2$,

$$x - 2 < 2 - (x - 1) \implies x - 2 < 3 - x \implies x < \frac{5}{2}$$

Thus,

$$x \in \left(\frac{1}{2}, \frac{5}{2}\right)$$

Equations and inequalities with absolute values – example 11

Question

Find the set of all real numbers that satisfy the inequality:

$$|x-2| < |x-4| - 1$$

If $x < 2$,

$$2 - x < (4 - x) - 1 \implies 2 - x < 3 - x \implies \text{always true}$$

If $2 \leq x < 4$,

$$x - 2 < 4 - x - 1 \implies x - 2 < 3 - x \implies 2x < 5 \implies x < \frac{5}{2}$$

If $x \geq 4$,

$$x - 2 < x - 4 - 1 \implies x - 2 < x - 5 \implies -2 < -5 \implies \text{false}$$

Thus, the solution is:

$$x \in \left(-\infty, \frac{5}{2}\right)$$

Equations and inequalities with absolute values – example 12

Question

Find the set of all real numbers that satisfy the inequality:

$$|x-3| > |x-1| - 1$$

If $x < 1$,

$$3 - x > (1 - x) - 1 \Longrightarrow 3 > 0 \Longrightarrow \text{always true}$$

If $1 \leq x < 3$,

$$3 - x > x - 1 - 1 \Longrightarrow x < \frac{5}{2}$$

If $x \geq 3$,

$$x - 3 > x - 1 - 1 \Longrightarrow x - 3 > x - 2 \Longrightarrow -3 > -2 \Longrightarrow \text{false}$$

Thus, the solution is:

$$x \in \left(-\infty, \frac{5}{2}\right)$$

Equations and inequalities with absolute values – example 13

Question

Find the set of all real numbers that satisfy the inequality:

$$|x-3|-1>|x-1|$$

If $x<1$,

$$3-x-1>1-x \implies 2>1 \implies \text{always true}$$

If $1\leq x<3$,

$$3-x-1>x-1 \implies x<\frac{3}{2}$$

If $x\geq 3$,

$$x-3-1>x-1 \implies -4>-1 \implies \text{false}$$

Thus, the solution is:

$$x\in\left(-\infty,\frac{3}{2}\right)$$

Equations and inequalities with absolute values – example 14

Question

Find the set of all real numbers that satisfy the equation:

$$|4 - 2x| + |3 + x| = 6.$$

Determine P.

If $x < -3$

$$4 - 2x - 3 - x = 6 \Longrightarrow x = -\frac{5}{3} \quad \text{(not valid)}$$

If $-3 \leq x < 2$

$$4 - 2x + 3 + x = 6 \Longrightarrow x = 1 \quad \text{(valid)}$$

If $x \geq 2$

$$2x - 4 + 3 + x = 6 \Longrightarrow x = \frac{7}{3} \quad \text{(valid)}$$

Thus, the solution set is:

$$\left\{1, \frac{7}{3}\right\}$$

Equations with absolute values – example 15

Question

Find the set of all real numbers that satisfy the equation:

$$|4 - 3x| + |3 + x| = 7.$$

If $x \leq -3$

$$4 - 3x - 3 - x = 7 \Longrightarrow x = -\frac{3}{2} \quad \text{(not valid)}$$

If $-3 < x \leq \frac{4}{3}$

$$4 - 3x + 3 + x = 7 \Longrightarrow x = 0 \quad \text{(valid)}$$

If $x > \frac{4}{3}$

$$3x - 4 + 3 + x = 7 \Longrightarrow x = 2 \quad \text{(valid)}$$

Thus, the solution set is:

$$\{0, 2\}$$

Equations with absolute values – example 16

Question

Find the set of all real numbers that satisfy the equation:

$$|1 - 3x| + |3 + x| = 4.$$

If $x < -3$

$$1 - 3x - 3 - x = 4 \implies x = -\frac{3}{2} \quad \text{(invalid)}$$

If $-3 \leq x < \frac{1}{3}$

$$1 - 3x + 3 + x = 4 \implies x = 0 \quad \text{(valid)}$$

If $x > \frac{1}{3}$

$$3x - 1 + 3 + x = 4 \implies x = \frac{1}{2} \quad \text{(valid)}$$

Thus, the solution set is:

$$\left\{0, \frac{1}{2}\right\}$$

Equations with absolute values – example 17

Question

Find the set of all real numbers that satisfy the equation:

$$|1 - 3x| + |4 + x| = 7.$$

If $x < -4$

$$1 - 3x - 4 - x = 7 \Longrightarrow x = -\frac{5}{2} \quad \text{(invalid)}$$

If $-4 \leq x < \frac{1}{3}$

$$1 - 3x + 4 + x = 7 \Longrightarrow x = -1 \quad \text{(valid)}$$

If $x > \frac{1}{3}$

$$3x - 1 + 4 + x = 7 \Longrightarrow x = 1 \quad \text{(valid)}$$

Thus, the solution set is:

$$\{-1, 1\}$$

8. Logarithmic equations and inequalities

Logarithmic equations and inequalities – example 1

Question

Solve

$$\log_2^2 x + 2\log_2 \sqrt{x} - 2 = 0,$$

where $x \in \mathbb{R}$.

Let $y = \log_2 x$, then

$$y^2 + y - 2 = 0 \Longrightarrow y = -2, 1 \Longrightarrow x = \frac{1}{4},\ 2$$

The logarithm $\log_2 x$ and $\log_2 \sqrt{x}$ are defined when:

$$x > 0$$

Both values are valid solutions.

Logarithmic equations and inequalities – example 2

Question

Solve

$$\log_3(4^x - 3) + \log_3(4^x - 1) = 1,$$

where $x \in \mathbb{R}$.

Let $y = 4^x$, then

$$\log_3(y-3) + \log_3(y-1) = 1 \Longrightarrow (y-3)(y-1) = 3 \Longrightarrow y = 0, 4$$

We have

$$x = 1$$

For the original expression to be defined, both logarithmic terms must have positive arguments:

$$4^x - 3 > 0 \Longrightarrow 4^x > 3 \Longrightarrow x > \log_4 3 \approx 0.792$$
$$4^x - 1 > 0 \Longrightarrow 4^x > 1 \Longrightarrow x > 0$$

$x = 1$ is a valid solution.

Logarithmic equations and inequalities – example 3

Question

Find all natural numbers that satisfy the following inequality

$$\frac{\log_{10}^2 x - 3\log_{10} x + 3}{\log_{10} x - 1} < 1$$

Let $y = \log_{10} x$, then

$$\frac{y^2 - 3y + 3}{y - 1} < 1 \Longrightarrow y < 1 \Longrightarrow x < 10$$

We have

$$x = 1, 2, 3, 4, 5, 6, 7, 8, 9$$

Domain constraint: $x > 0$, $x \neq 10$

Logarithmic equations and inequalities – example 4

Question

Find all integers that satisfy the following inequality

$$\log_{\frac{1}{2}}(1+2x) > -1$$

Note that

$$1+2x > 0 \implies x > -\frac{1}{2}$$

$$1+2x > \left(\frac{1}{2}\right)^{-1} \implies 1+2x < 2 \implies x < \frac{1}{2}$$

Thus

$$x = 0$$

Verify the domain constraints:

$$1+0 > 0$$

$x = 0$ is a valid solution.

Logarithmic equations and inequalities – example 5

Question

Solve

$$\log_3\left(7 + 10\log_2(x+1)\right) = 3,$$

where $x \in \mathbb{R}$.

$$7 + 10\log_2(x+1) = 27 \implies x + 1 = 4 \implies x = 3$$

Verify the domain constraints:

$$3 + 1 > 0$$

$$7 + 10\log_2(3+1) = 27 > 0$$

$x = 3$ is a valid solution.

Logarithmic equations and inequalities – example 6

Question

Solve

$$\log_4\left(6 + 5\log_3(x+2)\right) = 2,$$

where $x \in \mathbb{R}$.

$$6 + 5\log_3(x+2) = 16 \implies x + 2 = 3^3 \implies x = 7$$

Verify the domain constraints:

$$7 + 2 > 0$$

$$6 + 5\log_3(7+2) = 16 > 0$$

$x = 7$ is a valid solution.

Logarithmic equations and inequalities – example 7

Question

Solve

$$\log_5\left(9 + 4\log_2(x+4)\right) = 2,$$

where $x \in \mathbb{R}$.

$$9 + 4\log_2(x+4) = 25 \implies x + 4 = 2^4 \implies x = 12$$

Verify the domain constraints:

$$12 + 4 > 0$$

$$9 + 4\log_2(12+4) = 25 > 0$$

$x = 12$ is a valid solution.

Logarithmic equations and inequalities – example 8

Question

Solve

$$\log_3 \left(7 + 10 \log_5(x+1)\right) = 3,$$

where $x \in \mathbb{R}$.

$$7 + 10 \log_5(x+1) = 27 \implies x + 1 = 5^2 \implies x = 24$$

Verify the domain constraints:

$$24 + 1 > 0$$

$$7 + 10 \log_5(24+1) > 0$$

$x = 24$ is a valid solution.

Logarithmic equations and inequalities – example 9

Question

Solve

$$\log_3(4^x - 3) - \log_3\left(4^x - \frac{4}{3}\right) = 1,$$

where $x \in \mathbb{R}$.

$$\begin{aligned} 4^x - 3 &= 3 \times \left(4^x - \frac{4}{3}\right) = 3 \times 4^x - 4 \\ 2 \times 4^x &= 1 \\ x &= -\frac{1}{2} \end{aligned}$$

Verify the domain constraints:

$$4^x > 3 \implies x > \log_4(3) \approx 0.792, \quad 4^x > \frac{4}{3} \implies x > \log_4\left(\frac{4}{3}\right) \approx 0.207$$

Thus there is no real solution to this equation.

Logarithmic equations and inequalities – example 10

Question

Solve

$$\log_5(4^x + 1) - \log_5\left(4 - 2 \times 4^x\right) = 0$$

where $x \in \mathbb{R}$.

$$\begin{aligned} 4^x + 1 &= 4 - 2 \times 4^x \\ 3 \times 4^x &= 3 \\ x &= 0 \end{aligned}$$

$\log_5(4^x + 1)$ is defined for all real x

$$4 - 2 \times 4^x > 0 \Longrightarrow x < \log_4(2) = \frac{1}{2}$$

Solution $x = 0$ is valid.

Logarithmic equations and inequalities – example 11

Question

Solve

$$\log_4(3^x - 1) - \frac{1}{2}\log_2(5 - 3^x) = 0$$

where $x \in \mathbb{R}$.

$$\frac{\log_2(3^x - 1)}{\log_2 4} - \frac{1}{2}\log_2(5 - 3^x) = 0 \implies \log_2\left(\frac{3^x - 1}{5 - 3^x}\right) = 0$$

$$\frac{3^x - 1}{5 - 3^x} = 1 \implies x = 1$$

Verify the domain constraints:

$$3^x - 1 > 0 \implies x > 0$$

$$5 - 3^x > 0 \implies x < \log_3(5) \approx 1.46$$

Solution $x = 1$ is valid.

Logarithmic equations and inequalities – example 12

Question

Solve

$$\log_5\left(3^{x+1} - 2 \times 3^x - 2\right) = 2,$$

where $x \in \mathbb{R}$.

$$3^{x+1} - 2 \times 3^x - 2 = 25$$

Let $y = 3^x$, then

$$3y - 2y - 2 = 25 \implies y = 27 \implies x = 3$$

We must ensure the argument of the logarithm is positive:

$$3^4 - 2 \times 3^3 - 2 = 81 - 54 - 2 = 25 > 0$$

the domain condition is satisfied.

Logarithmic equations and inequalities – example 13

Question

Solve

$$\log_4 \left(2 + \log_2(x + 2)\right) = 1,$$

where $x \in \mathbb{R}$.

$$2 + \log_2(x + 2) = 4 \implies x = 2$$

Domain constraints:

$$2 + 2 > 0, \quad 2 + \log_2(2 + 2) > 0$$

Both conditions are satisfied for $x = 2$.

Logarithmic equations and inequalities – example 14

Question

Solve

$$\log_2(3x - 8) \leq 2$$

where $x \in \mathbb{Z}$.

$$3x - 8 \leq 4 \implies x \leq 4$$

Domain of the logarithm

$$3x - 8 > 0 \implies x > \frac{8}{3} \implies x \geq 3$$

Thus $x = 3,\ 4$

Logarithmic equations and inequalities – example 15

Question

Solve

$$\log_2(1+2x) < 1,$$

where $x \in \mathbb{Z}$.

$$1 + 2x < 2 \implies x < \frac{1}{2} \implies x \leq 0$$

Domain of the logarithm

$$1 + 2x > 0 \implies x > -\frac{1}{2} \implies x \geq 0$$

Thus $x = 0$

9. Exponential equations and inequalities

Exponential equations and inequalities – example 1

Question

Solve

$$2^{x-1} + 2^{x-2} + 2^{x-3} = 448,$$

where $x \in \mathbb{R}$.

Let $y = 2^{x-3}$, then

$$4y + 2y + y = 448 \implies y = 64 \implies x = 9$$

Exponential equations and inequalities – example 2

Question

Solve

$$3^{x+1} + 3^{2-x} = 28,$$

where $x \in \mathbb{R}$.

Let $y = 3^x$, then

$$3y + \frac{9}{y} = 28 \implies y = \frac{1}{3}, 9 \implies x = -1, 2$$

Exponential equations and inequalities – example 3

Question

Solve

$$4^x - 3^{x-\frac{1}{2}} = 3^{x+\frac{1}{2}} - 2^{2x-1},$$

where $x \in \mathbb{R}$.

Let $y = 3^{x-\frac{1}{2}}$ and $z = 2^{2x-1}$ then

$$2z - y = 3y - z \Longrightarrow 4y = 3z \Longrightarrow 3^{x-\frac{1}{2}} \times 4 = 3 \times 2^{2x-1} \Longrightarrow 3^{x-\frac{3}{2}} = 2^{2x-3}$$

We have

$$x - \frac{3}{2} = 2x - 3 = 0 \Longrightarrow x = \frac{3}{2}$$

Exponential equations and inequalities – example 4

Question

Solve

$$\left(\frac{2}{5}\right)^{x(x-3)} < \frac{25}{4}$$

where $x \in \mathbb{N}$.

$$\frac{25}{4} = \left(\frac{2}{5}\right)^{-2} \Longrightarrow x(x-3) > -2 \Longrightarrow x < 1 \text{ or } x > 2$$

Exponential equations and inequalities – example 5

Question

Solve

$$2^x - 4 \times 4^{x-1} > 4^{-1} - 2^{x-2}$$

where $x \in \mathbb{R}$.

Let $y = 2^{x-2}$, then

$$4y - 16y^2 > \frac{1}{4} - y \Longrightarrow \frac{1}{16} < y < \frac{1}{4} \Longrightarrow -2 < x < 0$$

Exponential equations and inequalities – example 6

Question

Solve

$$9^{x^2+5x+3} \geq 3^{x^2+4x-2}$$

where $x \in \mathbb{R}$.

$$2(x^2 + 5x + 3) \geq x^2 + 4x - 2 \Longrightarrow x \leq -4 \text{ or } x \geq -2$$

Exponential equations and inequalities – example 7

Question

Solve

$$4^{x^2+5x+4} \leq 2^{x^2+2x-7}$$

where $x \in \mathbb{Z}$.

$$2(x^2+5x+4) \leq x^2+2x-7 \Longrightarrow -5 \leq x \leq -3 \Longrightarrow x = -5, -4, -3$$

Exponential equations and inequalities – example 8

Question

Solve

$$4^{x^2+3x-4} = 16^{x^2+4x+1}$$

where $x \in \mathbb{Z}$.

$$x^2 + 3x - 4 = 2(x^2 + 4x + 1) \Longrightarrow x = -3, \ -2$$

Exponential equations and inequalities – example 9

Question

Solve

$$6^{x^2+4x-2} \leq 36^{x^2+5x+3}$$

where $x \in \mathbb{R}$.

$$x^2 + 4x - 2 \leq 2(x^2 + 5x + 3) \implies x \leq -4, \;\; x \geq -2$$

Exponential equations and inequalities – example 10

Question

Solve

$$25^{x^2+4x+1} \leq 5^{x^2+3x-4}$$

where $x \in \mathbb{R}$.

$$2(x^2 + 4x + 1) \leq x^2 + 3x - 4 \Longrightarrow -3 \leq x \leq -2$$

Exponential equations and inequalities – example 11

Question

Solve

$$2^{x-1} - 2^{x-2} + 2^{x-3} = 96$$

where $x \in \mathbb{R}$.

Let $y = 2^{x-3}$, then

$$4y - 2y + y = 96 \implies y = 32 \implies x = 8$$

Exponential equations and inequalities – example 12

Question

Solve

$$9^{x^2+4x+1} > 3^{x^2+6x+1}$$

where $x \in \mathbb{R}$.

$$2(x^2+4x+1) > x^2+6x+1 \implies (x+1)^2 > 0 \implies x \neq -1$$

Exponential equations and inequalities – example 13

Question

Solve

$$9^{x^2-2x+1} < 3^{x^2-x}$$

where $x \in \mathbb{R}$.

$$2(x^2 - 2x + 1) < x^2 - x \implies 1 < x < 2$$

Exponential equations and inequalities – example 14

Question

Solve

$$3^{x^2-4x+3} < 1$$

where $x \in \mathbb{R}$.

$$x^2 - 4x + 3 < 0 \Longrightarrow 1 < x < 3$$

Exponential equations and inequalities – example 15

Question

Solve

$$3^{x-1} - 3^{x+1} + 3^{x} = -405$$

where $x \in \mathbb{R}$.

Let $y = 3^{x-1}$

$$y - 9y + 3y = -405 \implies y = 81 \implies x = 5$$

Exponential equations and inequalities – example 16

Question

Solve

$$\left(\frac{7}{2}\right)^{x^2-4x+2} > \frac{4}{49}$$

where $x \in \mathbb{R}$.

$$x^2 - 4x + 2 > -2 \implies x \neq 2$$

Exponential equations and inequalities – example 17

Question

Solve

$$\left(\frac{7}{2}\right)^{x^2-4x} < 1$$

where $x \in \mathbb{R}$.

$$x^2 - 4x < 0 \Longrightarrow 0 < x < 4$$

10. Trigonometric equations

Trigonometric equations – example 1

Question

Solve

$$\cos(2x) = \cos^2(2x),$$

where

$$x \in \left(-\frac{\pi}{4}, \frac{8\pi}{5}\right).$$

$$\cos^2(2x) - \cos(2x) = 0 \Longrightarrow \cos(2x) = 0, 1 \Longrightarrow x = \frac{\pi}{4} + \frac{k\pi}{2},\ \ k\pi,\ \ k \in \mathbb{Z}$$

$$x = 0,\ \pm\frac{\pi}{4},\ \frac{3\pi}{4},\ \pi$$

Trigonometric equations – example 2

Question

Solve

$$\sin(x)\cos(x)\cos(2x) = -\frac{1}{2},$$

where $x \in \mathbb{R}$.

$$\begin{aligned} \frac{1}{2}\sin(2x)\cos(2x) &= -\frac{1}{2} \\ \frac{1}{2}\sin(4x) &= -1 \\ \sin(4x) &= -2 \end{aligned}$$

no solution

Trigonometric equations – example 3

Question

Solve

$$\sin(5x) - \sin(3x) = 0,$$

where

$$x \in \left(-\frac{\pi}{4}, \pi\right).$$

$$2\sin\left(\frac{5x-3x}{2}\right)\cos\left(\frac{5x+3x}{2}\right) = 0 \Longrightarrow 2\sin(x)\cos(4x) = 0$$

$$\sin(x) = 0 \text{ or } \cos(4x) = 0 \Longrightarrow x = k\pi, \text{ or } \frac{\pi}{8} + \frac{k\pi}{4}, \; k \in \mathbb{Z}$$

Finally,

$$x = -\frac{\pi}{8}, \; 0, \; \frac{\pi}{8}, \; \frac{3\pi}{8}, \; \frac{5\pi}{8}, \; \frac{7\pi}{8}$$

Trigonometric equations – example 4

Question

Solve

$$(\sin(x) - \cos(x))^2 = \sin(2x)$$

where

$$x \in \left(-\frac{11\pi}{12}, \frac{11\pi}{12}\right).$$

$$1 - 2\sin(x)\cos(x) = \sin(2x) \Longrightarrow 1 - \sin(2x) = \sin(2x) \Longrightarrow \sin(2x) = \frac{1}{2}$$

$$x = \frac{\pi}{12} + k\pi, \frac{5\pi}{12} + k\pi,\ k \in \mathbb{Z} \Longrightarrow x = -\frac{7\pi}{12}, \frac{\pi}{12}, \frac{5\pi}{12}$$

Trigonometric equations – example 5

Question

Solve

$$\cos(5x) = \cos(3x)$$

where

$$x \in \left[-\frac{\pi}{4}, \pi\right).$$

$$\cos(5x) - \cos(3x) = -2\sin\left(\frac{5x+3x}{2}\right)\sin\left(\frac{5x-3x}{2}\right) = -2\sin(4x)\sin(x) = 0$$

$$\sin(x) = 0 \text{ or } \sin(4x) = 0 \implies x = k\pi, \frac{k\pi}{4}, \quad k \in \mathbb{Z}$$

$$x = -\frac{\pi}{4}, \ 0, \ \frac{\pi}{4}, \ \frac{\pi}{2}, \ \frac{3\pi}{4}$$

Trigonometric equations – example 6

Question

Solve

$$(\sin(x) - \cos(x))^2 = \cos(2x)$$

where $x \in (-\pi, \pi]$.

$$\begin{aligned} \sin^2(x) + \cos^2(x) - 2\sin(x)\cos(x) &= \cos^2(x) - \sin^2(x) \\ 2\sin^2(x) - 2\sin(x)\cos(x) &= 0 \\ \sin(x)(\sin(x) - \cos(x)) &= 0 \end{aligned}$$

$$\sin(x) = 0 \text{ or } \sin(x) = \cos(x) \implies x = k\pi, \frac{\pi}{4} + k\pi, \ k \in \mathbb{Z}$$

$$x = -\frac{3\pi}{4}, \ 0, \ \frac{\pi}{4}, \ \pi$$

Trigonometric equations – example 7

Question

Solve

$$\sin(2x) - \sin^2(2x) = 0$$

where

$$x \in \left(-\frac{\pi}{4}, \frac{8\pi}{5}\right].$$

$$\sin(2x)(1 - \sin(2x)) = 0 \Longrightarrow \sin(2x) = 0, 1 \Longrightarrow x = \frac{k\pi}{2},\ \frac{\pi}{4} + k\pi,\ k \in \mathbb{Z}$$

$$x = 0,\ \frac{\pi}{4},\ \frac{\pi}{2},\ \pi,\ \frac{5\pi}{4},\ \frac{3\pi}{2}$$

Trigonometric equations – example 8

Question

Solve

$$\sin(x)\cos(x)\sin(2x) = \frac{\sqrt{3}}{2}$$

where $x \in \mathbb{R}$.

$$\begin{aligned} \frac{1}{2}\sin^2(2x) &= \frac{\sqrt{3}}{2} \\ \sin(2x) &= \pm 3^{\frac{1}{4}} \end{aligned}$$

Since $3^{\frac{1}{4}} > 1$, no solutions exist.

Trigonometric equations – example 9

Question

Solve

$$\sin(5x) - \sin(x) = 0,$$

where

$$x \in \left[-\frac{\pi}{4}, \pi\right).$$

$$\begin{aligned} 2\cos\left(\frac{5x+x}{2}\right)\sin\left(\frac{5x-x}{2}\right) &= 0 \\ 2\cos(3x)\sin(2x) &= 0 \end{aligned}$$

$\cos(3x) = 0$ or $\sin(2x) = 0 \Longrightarrow x = \dfrac{\pi}{6} + \dfrac{k\pi}{3}$ or $\dfrac{k\pi}{2}$, where $k \in \mathbb{Z}$

$$x = -\frac{\pi}{6},\ 0,\ \frac{\pi}{6},\ \frac{\pi}{2},\ \frac{5\pi}{6}$$

Trigonometric equations – example 10

Question

Solve

$$\sin(x) - \sin(2x) = 0,$$

where $x \in [0, \pi)$.

$$2\cos\left(\frac{3x}{2}\right)\sin\left(\frac{-x}{2}\right) = -2\cos\left(\frac{3x}{2}\right)\sin\left(\frac{x}{2}\right) = 0$$

$$\cos\left(\frac{3x}{2}\right) = 0 \text{ or } \sin\left(\frac{x}{2}\right) = 0 \Longrightarrow x = \frac{\pi}{3} + \frac{2k\pi}{3} \text{ or } 2k\pi, \text{ where } k \in \mathbb{Z}.$$

$$x = 0, \ \frac{\pi}{3}$$

Trigonometric equations – example 11

Question

Solve

$$\cos^2(x) + 2\sin(x) = 0,$$

where $x \in [0, \pi)$.

$$\begin{aligned} 1 - \sin^2(x) + 2\sin(x) &= 0 \\ \sin^2(x) - 2\sin(x) - 1 &= 0 \\ (\sin(x) - 1)^2 &= 2 \end{aligned}$$

$$\sin(x) = 1 \pm \sqrt{2}$$

No solutions exist.

Trigonometric equations – example 12

Question

Solve

$$\cos^2(2x) + 2\sin(2x) = 0,$$

where $x \in (0, \pi)$.

$$\begin{aligned} 1 - \sin^2(2x) + 2\sin(2x) &= 0 \\ \sin^2(2x) - 2\sin(2x) - 1 &= 0 \\ (\sin(2x) - 1)^2 &= 2 \end{aligned}$$

$$\sin(2x) = 1 \pm \sqrt{2}$$

No solutions exist.

Trigonometric equations – example 13

Question

Solve

$$\sin(2x) - \cos(x) = 0,$$

where

$$x \in \left[0, \frac{2\pi}{3}\right).$$

$$\begin{aligned} \sin(2x) &= \cos(x) \\ 2\sin(x)\cos(x) &= \cos(x) \end{aligned}$$

$$\cos(x) = 0 \text{ or } \sin(x) = \frac{1}{2} \implies x = \frac{\pi}{2} + k\pi, \text{ or } \frac{\pi}{6} + 2k\pi, \ \frac{5\pi}{6} + 2k\pi \text{ where } k \in \mathbb{Z}$$

$$x = \frac{\pi}{6}, \ \frac{\pi}{2}$$

Trigonometric equations – example 14

Question

Solve

$$\sin(x) - \cos(2x) = 0,$$

where $x \in [0, \pi)$.

$$\cos(2x) = 1 - 2\sin^2(x) \implies \sin(x) = 1 - 2\sin^2(x) \implies 2\sin^2(x) + \sin(x) - 1 = 0$$

$$\sin(x) = \frac{-1 \pm \sqrt{1^2 + 8}}{4} = \frac{-1 \pm 3}{4} \implies \sin(x) = \frac{1}{2},\ -1$$

$$x = \frac{\pi}{6} + 2k\pi,\ \frac{5\pi}{6} + 2k\pi \text{ or } -\frac{\pi}{2} + 2k\pi,\ \frac{3\pi}{2} + 2k\pi \text{ where } k \in \mathbb{Z}$$

$$x = \frac{\pi}{6},\ \frac{5\pi}{6}$$

Trigonometric equations – example 15

Question

Solve

$$\sin(x) + \cos(2x) = 0,$$

where $x \in [0, \pi)$.

$$\sin(x) + 1 - 2\sin^2(x) = 0 \implies 2\sin^2(x) - \sin(x) - 1 = 0$$

$$\sin(x) = \frac{1 \pm \sqrt{1+8}}{4} = \frac{1 \pm 3}{4} \implies \sin(x) = 1,\ -\frac{1}{2}$$

$$x = \frac{\pi}{2} + 2k\pi \text{ or } -\frac{\pi}{6} + 2k\pi,\ \frac{7\pi}{6} + 2k\pi \text{ where } k \in \mathbb{Z}$$

$$x = \frac{\pi}{2}$$